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Time Allocation Strategies for Entrepreneurial Operations Management

A dissertation submitted in partial satisfaction

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by

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To my parents

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ABSTRACT OF THE DISSERTATION

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This dissertation investigates three key operational issues that entrepreneurial firms encounter during their growth phases: (1) project selection via learning, (2) process improvement, and (3) hiring. In particular, we focus on entrepreneurial firms in an organizational life cycle in which the innovative product or service of the firm have found a market niche (and therefore has passed the survival phase), and the primary goal is to maximize growth. In this high-growth phase, the increasing number of tasks requiring the entrepreneur's attention places an overwhelming demand on the time of the entrepreneurs, who is in charge of all the decision making. Motivated by the theory of constraints, we provide insights to the above three key problems in operations management by examining the entrepreneur's time allocation decisions.

The first essay examines the project selection problem under uncertainty and learning leadtime, modeled by the discrete time bandit problems with stochastic response delays. It provides theoretical contribution to the extant restless bandit literature by proving that such class of bandit problems satisfies the indexability criterion as long as the delayed responses do not cross over. Thus, the problem is made practically solv-

able (near optimally) by employing the resulting marginal productivity index (MPI)-based heuristic. The result holds for infinite or finite horizon and holds for arbitrary delay lengths and infinite state spaces. We compute the resulting MPI's for the Beta-Bernoulli Bayesian delayed learning model, formulate and compute a tractable upper bound, and numerically validate the MPI-policy's near optimal performance.

The second essay investigates how entrepreneurial firms should invest time in process improvement decisions during growth. For many entrepreneurial firms during the growth phase, their main bottleneck resource is the entrepreneur's time, rather than cash. We classify an entrepreneur's daily activities into four categories: fire-fighting (spending time to attend to random urgent disruptions), process improvement (investing time to reduce future fire-fighting frequency), revenue enhancement (investing time to enhance the revenue stream), and revenue generation (spending time to harvest revenue at the prevailing rate), and analyze a stylized dynamic time allocation problem for maximizing long-term expected profits. We find that entrepreneurs should first invest time in process improvement until the process reliability reaches a certain threshold, then in revenue enhancement until the revenue rate reaches a certain threshold, and only then spend time generating revenue. Moreover, the greater the relative growth opportunities the entrepreneur foresee, more time should be spent upfront on process improvement to create a greater upfront safety-stock of time which can be used during growth. Furthermore, we find that entrepreneurs with higher prevailing revenue rate encounter the tradeoff between investing time in process improvement and investing time in revenue-related activities earlier than their counterparts with lower prevailing revenue rates, which leads them to settle at a lower process reliability and revenue rate than their counterparts. Thus, while they invest time optimally, the leader will ultimately lose their role as the leader, suggesting that it is not necessarily the complacency of the leader that causes the leader and follower reversal.

The third essay presents a formalized model of the entrepreneurial production and provides insights into the entrepreneurial firms' hiring decision by examining how the inputs of time and money interact. Entrepreneurs' time and money are two key complementary inputs for any entrepreneurial firm's production, and the lack of either resource constrains the firm's growth. We demonstrate that the shadow value of time always becomes greater than the shadow value of money, making time the key bottleneck resource. Viewing hiring as an opportunity for trading off money against time, we characterize the optimal timing of the hiring decisions faced by entrepreneurial firms. We establish that there is a unique cash level threshold above which it is optimal to hire. We find that this hiring threshold is non-monotonic in the hiring setup time, due to the tradeoff between the need to preserve the growth momentum and the need to hire before the shadow value of time becomes too large. On the other hand, entrepreneurs should delay hiring if the setup cost increases, suggesting the importance of differentiating setup cost and setup time in the hiring decisions. Finally, we find that the optimal timing of hiring maximizes (rather than minimizes) the post-hire shadow value gap between time and money.

CHAPTER I

Indexability of Bandit Problems with Response Delays

1 Introduction

1.1 Motivation

Dynamic allocation of activity under uncertainty is a fundamental decision problem faced by decision makers everyday. In each time period, unable to engage in (pull) all of the existing projects (arms), the decision maker must carefully choose a subset of the projects to engage in. Once the projects are chosen, corresponding events are set in motion, outcomes are observed, based on which the states of each projects are updated. The objective of the decision maker is to utilize the given information about the projects and choose the subset of projects each period that would maximize the long term horizon discounted rewards.

This widely studied dynamic decision model, a variant of a problem better known as the multiarmed bandit problem, however, ignores an important dimension of response delays. In practice, a project's outcome is not observed immediately, but only after a *delay* (whose length may be random), during which the decision maker continues to make decisions. Incorporating delays provide a powerful modeling framework, as it can be generalized to aid decision making in many application areas. We illustrate a few examples.

- **Clinical Trials (Whittle 1988).** In this setting, the arms correspond to medical treatments. The state of an arm represents one's state of knowledge on the effectiveness of the corresponding treatment. Pulling an arm corresponds to treating a patient with the corresponding medical treatment. One's state of knowledge on the effectiveness of the treatment will be updated only after observing the patient's treatment outcome.

- **Dynamic Assortment (Caro and Gallien 2007).** In this setting, the set of arms represent the unproduced assortment of fashion items. The state of each project represents one's knowledge on how popular the item will be. The knowledge of each item's popularity will be refined only after observing the sales, which is possible only after incurring production and distribution leadtime.

- **Corporate Strategy (Bernardo and Chowdhry 2002).** In this setting, the arms correspond to regions where franchises can be opened. The state of the arms represents the revenue expectations of each region prior to opening a franchise. Once a franchise is opened, the actual sales is observed only after a delay during which the franchise reaches out to the customers. During the delay, the headquarters may decide to open more franchises in the region.

- **Management of Employees.** In this setting, the arms represent employees, and the state of the arms represents the manager's belief about the skill level of each employee. After delegating assignments to different employees, the manager can update his belief on each employee's skill levels based on the outputs, which occurs only after a delay.

Despite their practical relevance, the bandit problems with response delays have received only moderate attention in the literature (see §1.2 for a review). One reason is because the problem becomes an intractable *restless bandit* problem (Whittle 1988), as the state of an arm which is not pulled (passive) may still change when a backlogged decision is implemented.

While they are difficult to solve optimally, many restless bandit problems can nonetheless be solved near optimally using the marginal productivity index (MPI)-

based heuristic (Niño-Mora 2006), provided that the problem satisfies the *indexability criterion* (Whittle 1988). Hence, indexability is a desirable property as it makes the restless bandit problem practically solvable by employing the MPI-based heuristic.

In this paper, we prove that the discrete time bandit problems with stationary random delays satisfy the *indexability criterion* as long as the delayed responses do not crossover. After an overview of the related literatures in §1.2, we introduce the multiarmed bandit problem with response delay and describe the basic properties in §2. In §3, we present the indexability result, and in §4, we compute the indices for the multiarmed bandit with delay for the canonical Beta-Bernoulli learning model and test its performance and compare it to those of other closed-form index heuristics. We conclude in §5.

1.2 Literature Review

The literature on restless bandit indexation was created when Whittle (1988) first generalized the classic bandit framework (Gittins 1979) by allowing the passive arms to change states, and termed it the *restless bandit* problem. The restless bandit problems are computationally intractable to solve optimally, and hence the primary research concerns the development of heuristic policies that can be shown to be near optimal. Whittle forms a Lagrangian dual problem and defines a priority index as the Lagrange multiplier associated with an arm which makes the decision maker indifferent between pulling and not pulling the arm. He shows that this priority index generalizes the Gittins index and devises a priority-index policy which pulls the arms with the highest index values. He further conjectures the asymptotic optimality of the priority-index policy, which Weber and Weiss (1990) later largely validate and Weiss (1992) shows a special case for which the conjecture holds. However, Whittle states that for the index to be well-defined, the restless bandit problem must first satisfy the *indexability criterion*. That is, the Lagrange multiplier that equates the pulling and non-pulling actions must be unique for every possible state of a given arm. He shows that indexibility cannot be taken for granted by providing counterexamples. Moreover, verifying indexibility itself is non-trivial and until recently sufficient conditions satisfied by a broad subclass of restless bandits were unknown.

Niño-Mora pioneers the field of restless bandit indexation to theoretically provide sufficient conditions for indexability. In particular, Niño-Mora (2006) generalizes the Whittle's priority index by defining the *marginal productivity index* (MPI) in terms of the more general and economically intuitive reward/work measure, and shows that the MPI's interpretation can be applied in an identical manner to many other classic index policies that were shown to be optimal, including the celebrated Gittins index. Using the MPI, he identifies classes of restless bandit problems that satisfy the sufficient conditions, mostly under the assumption of a finite state space (see Niño-Mora 2007 and references therein). Our work contributes to the literature by expanding the known class of indexable restless bandit problems.

The problem of bandits with response delays has received only a moderate attention in the literature. Eick (1988) examines the clinical trials setting where a patient's lifetime is modelled as a geometric random variable, and provides the first proof of indexability for a delayed response bandit when the discount factor δ is than $1/2$. Wang and Bickis (2003) extend this result to arbitrary lifetime distributions under certain regularity conditions, but those conditions reduce to $\delta < 1/2$ in the discrete time case. In contrast, our result shows indexability for the more applicable discount factors $\delta < 1$. Hardwick et al. (2006) consider the response delay bandit model where patients arrive according to a Poisson process with the treatment time having exponential response delays. They identify heuristics that perform well under the objective of minimizing patient loss. However, the heuristics are randomized rules which are not grounded in indexability theory. More recently, Niño-Mora (2007) examines a finite queue with a one period response delay and shows its indexability. However, the model lacks generality in that the state space must be finite and the delay is limited to one period, whereas our model allows for infinite state space and arbitrary delay lengths, which can be stationary random as long as the delayed responses do not crossover. We refer the interested reader to Altman and Stidham (1995) and Ehsan and Liu (2004) for other queueing applications with delayed information. Finally, Caro and Galien (2007) introduces a closed-form index, generalizes it to incorporate response delay, and show that the resulting index policy has near-optimal performance. Our work suggests that their method performs well because their closed-form index is a good approximation of the MPI.

2 Problem Description

2.1 Model Basics

The decision problem is defined in discrete time, where each period is indexed by t, representing t steps to go, and the rewards are discounted by $\delta < 1$ each period. The response delay ℓ is also a discrete quantity. In each time period, with S available arms but only able to pull N ($N < S$), the decision maker must carefully assess the state of each arm s. Once the arms are pulled, the outcomes are observed ℓ-period later, at which point the state of the arm changes. The objective of the decision maker is to pull the N arms each period to maximize the long term discounted rewards.

Let $x_s \in \Re$ denote the state of arm s and the vector $\mathbf{x} \in \Re^S$ denote the state of all S

arms. Let $R_s(x_s)$ denote the reward of arm s which depends on its state. For simplicity, we assume that the reward functions R_s are uniformly bounded, but this assumption can be relaxed (for instance, see Condition B in p. 17 of Gittins 1989 or the Bayesian formulation given in Burnetas and Katehakis 2003). The decision on arm s each period is represented by $u_s \in \{0,1\}$, where a value of $u_s = 1$ corresponds to a $(Pull)$ decision, while a value of $u_s = 0$ corresponds to a $(NotPull)$ decision. In each period, it is not possible to pull more than N arms, i.e. $\sum_s^S u_s \leq N$. The vector $\mathbf{u} \in \{0,1\}^S$ denote the decision on all S arms, and each of the vectors $(\mathbf{v}^1,...,\mathbf{v}^\ell)$ represent the decisions that had been made in previous periods, with $\mathbf{v}^1$ being the oldest decision that will be implemented this period and the $\mathbf{v}^\ell$ being the most recent decision.

Each arm s follows an independent Markovian process. If $v_s^1 = 1$, the function $f_s(x_s, v_s^1, w_s)$ denotes the state that the arm s transitions to from state x_s given the decision v_s^1 and the random component $w_s(x_s)$, which depends on state x_s; and if $v_s^1 = 0$, $f_s(x_s, v_s^1, w_s) = x_s$ signifying that the state of the arm remains unchanged. Letting the vector $\mathbf{w(x)} \in \Re^S$ represent a vector of random variables $w_s(x_s)$, the vector $f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}) \in \Re^S$ represents state that all the arms transitions to from state $\mathbf{x}$ given the decision vector $\mathbf{v}^1$ and the random component $\mathbf{w}$.

Let $\mathbf{J}_t^*(\mathbf{x}, \mathbf{v}^1,...,\mathbf{v}^\ell)$ denote the maximum discounted reward with t steps to go given the state of the arms $\mathbf{x}$ and the decisions of the previous periods $\mathbf{v}^1,\ldots,\mathbf{v}^\ell$. Then, the multiarmed bandit problem with delay can be expressed as the following dynamic program,

$$\text{(BD):} \quad \mathbf{J}_t^*(\mathbf{x}, \mathbf{v}^1,...,\mathbf{v}^\ell) = \max_{\mathbf{u}}\{\textstyle\sum_{s=1}^S R_s(x_s)v_s^1 + \delta \mathrm{E}_{\mathbf{w}} \mathbf{J}_{t-1}^*(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2,...,\mathbf{v}^\ell, \mathbf{u})\}$$
$$\text{s.t.} \quad \mathbf{u} \in \{0,1\}^S,\ \textstyle\sum_{s=1}^S u_s \leq N$$

for $t > \ell$, and for $t \leq \ell$,

$$\mathbf{J}_t^*(\mathbf{x}, \mathbf{v}^1, ..., \mathbf{v}^\ell) = \quad \sum_{s=1}^{S} R_s(x_s) v_s^1 + \delta \mathrm{E}_{\mathbf{w}} \mathbf{J}_{t-1}^*(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, ..., \mathbf{v}^\ell)\}, \quad \mathbf{J}_0^*(\cdot) \equiv 0.$$

Because a passive arm's state can change when the delayed (*Pull*) decision is implemented (i.e. $v_s^1 = 1$), the problem is a restless bandit problem (Whittle 1988), which is intractable. A known heuristic which can solve the restless bandit problems near optimally is the MPI-index policy, but this policy is well-defined only if the problem satisfies the indexability criteria. We explain this next.

2.2 Indexability Criterion and the Equivalence Relation

Whittle (1988) forms the Lagrangian dual of the restless bandit problem, where the dual variable λ has the interpretation of *subsidy* for not pulling the arm, and defines the priority index as the value of λ that makes the decision indifferent between pulling and not pulling the arm. If the index λ is to be meaningful however, it must induce a consistent ordering of the arms, in that any arm which is not pulled under a subsidy λ will also be not pulled under a higher subsidy $\lambda' > \lambda$. An equivalent statement in terms of *cost* is that if any arm is pulled under a cost λ, it must also be pulled under a lower cost $\lambda' < \lambda$. The formal definition of indexability for a single independent arm is the following:

Definition (Whittle 1988). *Let $D_s(\lambda)$ be the set of values of x_s for which project s would be rested under a λ-subsidy policy. Then the project is* indexable *if $D_s(\lambda)$ increases monotonically from $\emptyset$ to Υ_s as λ increases from $-\infty$ to $+\infty$, where Υ_s is the full state space for project s.*

A restless bandit problems is indexable if each one of its arms is indexable. The proof given in the next section shows that a single-arm bandit with response delay satisfies Whittle's definition. Therefore, the multiarmed bandit problem with response delays (BD) is indexable.

Before showing the main indexibility result, note that there are potentially four possible formulations of our problem, due to the four different ways of accounting for the Lagrange multiplier λ in the rewards. First, from the definition, λ can either have a *subsidy* or *cost* interpretation. Moreover, it may be accounted for when the pull/not pull decision is *made* (before the delay) or when the decision gets *implemented* (after the delay). Accounting for λ before the delay has been more prevalent in the literature. For instance, Wang and Bickis (2003) and Caro and Gallien (2007) consider the subsidy and cost interpretations respectively under that framework. In our proof of indexibility we found it easier to account for λ after the delay. Regardless of this choice, our first proposition shows that the accounting method does not affect the indexability result. Furthermore, because the order of the indices do not change, the priority index policy whose indices are derived from four different accounting methods would be identical.

Proposition I.1 *Suppose one formulation is indexable. Then the other three formulations are also indexable. Moreover, the ranking of the indices does not change from one formulation to the other.*

Proof. See Appendix A.

In the next section, without a loss of generality, we examine the formulation where λ represents a subsidy for not pulling and rewards (including the subsidy) are accounted when the decisions are implemented after the delay.

3 Structural Results

In this section, we establish the indexability of multiarmed bandit problem with (i) constant delay, and then (ii) with stationary random delays in which the delayed responses do not cross over. We do so by showing that the single-arm bandit with response delay is indexable.

We point out that the underlying bandit problem is not restless, or in other words the state of arm that is not pulled does not change ℓ periods later. Only after the incorporation of the response delay does the problem become 'restless.' We exploit this underlying non-restless structure of the problem in the proof by matching sample paths.

Let $J_{t,s}^{\lambda}(z)$ denote the maximum profit-to-go function of an arm s with a subsidy λ for a state z with t periods to go. To show that arm s satisfies Whittle's indexibility definition, for each state z a unique index λ must exist such that the expected discounted profit from pulling is equal to that from not pulling. More formally, if we denote the maximum profit-to-go function of the arm after it is pulled and after it is not pulled respectively as

$$J_{t,s}^{\lambda}(z)^{(Pull)}, \quad \text{and} \quad J_{t,s}^{\lambda}(z)^{(NotPull)},$$

we would want to show that there exists a unique λ such that $J_{t,s}^{\lambda}(z)^{(Pull)} = J_{t,s}^{\lambda}(z)^{(NotPull)}$. We can achieve this if we show that $\Delta J_{t,s}^{\lambda}(z) \equiv J_{t,s}^{\lambda}(z)^{(Pull)} - J_{t,s}^{\lambda}(z)^{(NotPull)}$ is a decreasing function of λ for every state z, and then take the limit when $t \longrightarrow \infty$.

3.1 Indexability of Constant Delay

Consider a single-arm s and a constant response delay of ℓ periods. The maximum profit-to-go function at time t and state x_s with delayed orders $(v_s^1, \ldots, v_s^\ell)$ is given by

$$\begin{aligned} J_{t,s}^\lambda(x_s, v_s^1, \ldots, v_s^\ell) &= R_s(x_s)v_s^1 + \lambda(1 - v_s^1) \\ &\quad + \delta \max\{E_{w_s} J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 1), \\ &\qquad E_{w_s} J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 0)\}. \end{aligned}$$

The expectation is taken with respect to the random variable w_s, which has an arbitrary distribution that is dependent on the current state x_s. When necessary, we will write $w_s(x_s)$ to make the parameter dependence explicit.

The difference in value at time t between the $(Pull)$ and $(NotPull)$ decisions has the following expression:

$$\begin{aligned} \Delta J_{t,s}^\lambda(x_s, v_s^1, \ldots, v_s^\ell) &= R_s(x_s)v_s^1 + \lambda(1 - v_s^1) + \delta E_{w_s} J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 1) \\ &\quad - \{R_s(x_s)v_s^1 + \lambda(1 - v_s^1) + \delta E_{w_s} J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 0)\} \\ &= \delta E_{w_s}\{J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 1) \\ &\quad - J_{t-1,s}^\lambda(f_s(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^\ell, 0)\}. \end{aligned}$$

Letting $z_s = (x_s, v_s^1, \ldots, v_s^\ell)$ denote the augmented state, we can rewrite the value function as,

$$J_{t-1,s}^\lambda(z_s) = J_{t-1,s}^\lambda(z_s)^{(Pull)} + [\Delta J_{t-1,s}^\lambda(z_s)]^- = J_{t-1,s}^\lambda(z_s)^{(NotPull)} + [\Delta J_{t-1,s}^\lambda(z_s)]^+,$$

where $[r]^+ = \max\{0, r\}, \quad [r]^- = \max\{0, -r\}$.

We now prove the monotonicity result. The key step of the proof uses a coupling argument to show the desired inequality. For notational simplicity, we will omit the subscript s in the proof.

Proposition I.2 *For all augmented state z, $\Delta J_t^{\lambda}(z)$ is decreasing in λ.*

Proof. Using induction, we show that for any $\lambda_1 > \lambda_2$, $\Delta J_t^{\lambda_1}(z) < \Delta J_t^{\lambda_2}(z)$ for all z. The proof is for $\ell \geq 2$. For $\ell = 1$ the notation would have to be slightly different but the argument is exactly the same.

Base Case: $t = \ell + 1$.

Here, we make the $(Pull)/(NotPull)$ decision only once, and observe the expected outcome in the remaining ℓ periods. We have,

$$\begin{aligned}
\Delta J_{\ell+1}^{\lambda}(x, v^1, \ldots, v^{\ell}) &= \delta E_{w_1}\{J_{\ell}^{\lambda}(f(x, v^1, w), v^2, \ldots, v^{\ell}, 1) - J_{\ell}^{\lambda}(f(x, v^1, w), v^2, \ldots, v^{\ell}, 0)\} \\
&= \delta^{\ell} E_{w_1} E_{w_2} \ldots E_{w_l}\{J_1^{\lambda}(f^{\circ\ell}(x, \underline{v}, \underline{w}), 1) - J_1^{\lambda}(f^{\circ\ell}(x, \underline{v}, \underline{w}), 0)\} \\
&= \delta^{\ell}\{E_{\underline{w}} R(f^{\circ\ell}(x, \underline{v}, \underline{w})) - \lambda\},
\end{aligned}$$

where the vector $\underline{v}$ represents all the delayed decisions $(v^1, \ldots, v^{\ell})$, the vector $\underline{w}$ represents the series of dependent random variables $(w_1, w_2, \ldots, w_{\ell})$, and $f^{\circ\ell}(x, \underline{v}, \underline{w})$ is a short-hand notation for $f(f \cdots f(f(f(x, v^1, w_1), v^2, w_2), v^3, w_3), \ldots), v^{\ell}, w_{\ell})$. Each w_i's distribution depends on the sample path of the states, and the expression $E_{\underline{w}}$ represents an ℓ-iterated expectation framework. This expression is clearly decreasing in λ, $\forall z$.

Induction Step: $t > \ell + 1$.

Assume that $\forall z = (x, v^1, \ldots, v^{\ell})$ and $\lambda_1 > \lambda_2$, $\Delta J_{t-1}^{\lambda_1}(z) < \Delta J_{t-1}^{\lambda_2}(z)$. We will show that $\forall z = (x, v^1, \ldots, v^{\ell})$ and $\lambda_1 > \lambda_2$, $\Delta J_t^{\lambda_1}(z) < \Delta J_t^{\lambda_2}(z)$.

We write out the expression for $\Delta J_{t-1}^{\lambda_1}(z)$ and $\Delta J_{t-1}^{\lambda_2}(z)$ as follows:

$$\begin{aligned}
\Delta J_t^{\lambda_1}(x,v^1,...,v^\ell) &= \delta E_{w_1}\{J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,1) \\
&\quad -J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,0)\}, \\
\Delta J_t^{\lambda_2}(x,v^1,...,v^\ell) &= \delta E_{w_1'}\{J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,1) \\
&\quad -J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,0)\}.
\end{aligned}$$

The difference between the first and second expressions gives us the following:

$$\begin{aligned}
\text{DIFF} &\equiv \Delta J_t^{\lambda_1}(x,v^1,...,v^\ell) - \Delta J_t^{\lambda_2}(x,v^1,...,v^\ell) \\
&= \delta E_{w_1}\{J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,1) - J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,0)\} \\
&\quad -(\delta E_{w_1'}\{J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,1) - J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,0)\}).
\end{aligned}$$

After rewriting each row's expression $J = \max\{J^{(Pull)}, J^{(NotPull)}\}$ in terms of $J = J^{(Pull)} + [J^{(NotPull)} - J^{(Pull)}]^+$ and $J = J^{(NotPull)} + [J^{(NotPull)} - J^{(Pull)}]^-$ and rearranging the terms, we have,

$$\begin{aligned}
\text{DIFF} &= \delta E_{w_1}\{J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,1)^{(NotPull)} \\
&\qquad -J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,0)^{(Pull)}\} \\
&\quad -\delta E_{w_1'}\{J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,1)^{(NotPull)} \\
&\qquad -J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,0)^{(Pull)}\} \\
&\quad +\delta E_{w_1}[\Delta J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,1)]^+ \\
&\qquad -\delta E_{w_1'}[\Delta J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,1)]^+ \\
&\quad -\delta E_{w_1}[\Delta J_{t-1}^{\lambda_1}(f(x,v^1,w_1),v^2,...,v^\ell,0)]^- \\
&\qquad +\delta E_{w_1'}[\Delta J_{t-1}^{\lambda_2}(f(x,v^1,w_1'),v^2,...,v^\ell,0)]^-.
\end{aligned}$$

Each of the last two rows is less than or equal to 0 via the induction assumption, and we will denote the sum of the last two rows as $C \leq 0$. After evaluating out each term in the first two rows, e.g.,

$$J^{\lambda_1}_{t-1}(f(x,v^1,w_1),v^2,...,v^\ell,1)^{(NotPull)} = R(f(x,v^1,w_1))v^2 + \lambda_1(1-v^2)$$
$$+\delta E_{w_2} J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0),$$

we arrive at the following expression:

$$\begin{aligned}
\text{DIFF} &= \delta E_{w_1}\{R(f(x,v^1,w_1))v^2 + \lambda_1(1-v^2) \\
&\qquad +\delta E_{w_2} J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0)\} \\
&\quad -\delta E_{w_1}\{R(f(x,v^1,w_1))v^2 + \lambda_1(1-v^2) \\
&\qquad +\delta E_{w_2} J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,0,1)\} \\
&\quad -\delta E_{w'_1}\{R(f(x,v^1,w'_1))v^2 + \lambda_2(1-v^2) \\
&\qquad +\delta E_{w'_2} J^{\lambda_2}_{t-2}(f(f(x,v^1,w'_1),v^2,w'_2),v^3,...,v^\ell,1,0)\} \\
&\quad +\delta E_{w'_1}\{R(f(x,v^1,w'_1))v^2 + \lambda_2(1-v^2) \\
&\qquad +\delta E_{w'_2} J^{\lambda_2}_{t-2}(f(f(x,v^1,w'_1),v^2,w'_2),v^3,...,v^\ell,0,1)\} + C \\
&\leq \delta^2 E_{w_1} E_{w_2}\{J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0) \\
&\qquad -J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,0,1)\} \\
&\quad +\delta^2 E_{w'_1} E_{w'_2}\{J^{\lambda_2}_{t-2}(f(f(x,v^1,w'_1),v^2,w'_2),v^3,...,v^\ell,0,1) \\
&\qquad -J^{\lambda_2}_{t-2}(f(f(x,v^1,w'_1),v^2,w'_2),v^3,...,v^\ell,1,0)\}.
\end{aligned}$$

We now introduce the coupling argument. Consider the bandit with subsidy λ_1 starting from two different states. The first, which we refer to as System A with time $t-2$ to go from state $f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0)$ and follows the optimal policy. The second, which we refer to as System B, starts from state

$f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,0,1)$, but it implements the same decision as System A in the first ℓ stages, and after that, it follows its own optimal policy. Let π^* denote the optimal policy of System A (which is followed by System B for the first ℓ periods). Note that both System A and System B start from the same (non-augmented) state $f(f(x,v^1,w_1),v^2,w_2)$ and experience the same number of state transitions within the next ℓ periods. Moreover, these transitions have exactly the same Markovian dynamics, so by defining the two processes on a common probability space, we can assume that the actual transitions are the same.

Let $G^{\lambda_1}_{\pi^*_{t-2}}(z)$ represent the value of being in state z with time $t-2$ to go and following the policy π^*. Then we have,

$$J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0) = G^{\lambda_1}_{\pi^*_{t-2}}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,1,0),$$

and

$$J^{\lambda_1}_{t-2}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,0,1) \geq G^{\lambda_1}_{\pi^*_{t-2}}(f(f(x,v^1,w_1),v^2,w_2),v^3,...,v^\ell,0,1).$$

The first equality and the second inequality follow because π^* is optimal for System A but suboptimal for System B. The same coupling argument can be used for the bandit with subsidy λ_2, only that System A would start in state (...,0,1) and System B would start in (...,1,0). Denoting π^{**} as the optimal policy of System A, we have,

$$J^{\lambda_2}_{t-2}(f(f(x,v^1,w_1'),v^2,w_2'),v^3,...,v^\ell,0,1) = G^{\lambda_2}_{\pi^{**}_{t-2}}(f(f(x,v^1,w_1'),v^2,w_2'),v^3,...,v^\ell,0,1),$$

and

$$J^{\lambda_2}_{t-2}(f(f(x,v^1,w_1'),v^2,w_2'),v^3,...,v^\ell,1,0) \geq G^{\lambda_2}_{\pi^{**}_{t-2}}(f(f(x,v^1,w_1'),v^2,w_2'),v^3,...,v^\ell,1,0).$$

By subtracting these smaller values of $G^{\lambda_2}_{\pi^{**}_{t-2}}$ and $G^{\lambda_1}_{\pi^*_{t-2}}$, the DIFF can be bounded above as follows:

$$
\begin{aligned}
\text{DIFF} \le \quad & \delta^2 E_{w_1} E_{w_2} \{ G^{\lambda_1}_{\pi^*_{t-2}} (f(f(x,v^1,w_1),v^2,w_2),v^3,\dots,v^\ell,1,0) && \text{(DIFF.1)} \\
& \qquad\qquad - G^{\lambda_1}_{\pi^*_{t-2}} (f(f(x,v^1,w_1),v^2,w_2),v^3,\dots,v^\ell,0,1) \} \\
& + \delta^2 E_{w'_1} E_{w'_2} \{ G^{\lambda_2}_{\pi^{**}_{t-2}} (f(f(x,v^1,w'_1),v^2,w'_2),v^3,\dots,v^\ell,0,1) && \text{(DIFF.2)} \\
& \qquad\qquad - G^{\lambda_2}_{\pi^{**}_{t-2}} (f(f(x,v^1,w'_1),v^2,w'_2),v^3,\dots,v^\ell,1,0) \}
\end{aligned}
$$

We now evaluate the expressions (DIFF.1) and (DIFF.2).

(DIFF.1):

$$
\begin{aligned}
& \delta^2 E_{w_1} E_{w_2} G^{\lambda_1}_{\pi^*_{t-2}} (f(f(x,v^1,w_1),v^2,w_2),v^3,\dots,v^\ell,1,0) \\
& \quad = \delta^\ell E_{w_1} E_{w_2} \dots E_{w_\ell} G^{\lambda_1}_{\pi^*_{t-l}} (f^{\circ\ell}(x,\underline{v},\underline{w}),1,0,u^*_1,\dots,u^*_{\ell-2}) \\
& \quad = \delta^\ell E_{w_1} E_{w_2} \dots E_{w_\ell} \{ R(f^{\circ\ell}(x,\underline{v},\underline{w})) \\
& \qquad\quad + \delta E_{w_{\ell+1}} G^{\lambda_1}_{\pi^*_{t-\ell-1}} (f(f^{\circ\ell}(x,\underline{v},\underline{w}),1,w_{\ell+1}),0,u^*_1,\dots,u^*_{\ell-2},u^*_{\ell-1}) \} \\
& \quad = \delta^\ell E_{\underline{w}} \{ R(f^{\circ\ell}(x,\underline{v},\underline{w})) \\
& \qquad\quad + \delta E_{w_{\ell+1}} (\lambda_1 + \delta J^{\lambda_1}_{t-\ell-2} (f(f^{\circ\ell}(x,\underline{v},\underline{w}),1,w_{\ell+1}),u^*_1,\dots,u^*_{\ell-1},u^*_\ell)) \},
\end{aligned}
$$

where u^*_τ means the optimal τ-th action for System A. Similarly,

$$
\begin{aligned}
& \delta^2 E_{w_1} E_{w_2} G^{\lambda_1}_{\pi^*_{t-2}} (f(f(x,v^1,w_1),v^2,w_2),v^3,\dots,v^\ell,0,1) \\
& \quad = \delta^\ell E_{w_1} E_{w_2} \dots E_{w_\ell} G^{\lambda_1}_{\pi^*_{t-l}} (f^{\circ\ell}(x,\underline{v},\underline{w}),0,1,u^*_1,\dots,u^*_{\ell-2}) \\
& \quad = \delta^\ell E_{w_1} E_{w_2} \dots E_{w_\ell} \{ \lambda_1 + \delta G^{\lambda_1}_{\pi^*_{t-\ell-1}} (f^{\circ\ell}(x,\underline{v},\underline{w}),1,u^*_1,\dots,u^*_{\ell-2},u^*_{\ell-1}) \} \\
& \quad = \delta^\ell E_{\underline{w}} \{ \lambda_1 + \delta (R(f^{\circ\ell}(x,\underline{v},\underline{w})) \\
& \qquad\quad + \delta E_{w_{\ell+2}} J^{\lambda_1}_{t-\ell-2} (f(f^{\circ\ell}(x,\underline{v},\underline{w}),1,w_{\ell+2}),u^*_1,\dots,u^*_{\ell-1},u^*_\ell)) \}.
\end{aligned}
$$

Both $w_{\ell+1}$ and $w_{\ell+2}$ have dependence on the same state $f^{\circ\ell}(x,\underline{v},\underline{w})$ and hence have the same distribution. Therefore, the last terms from the expressions, $\delta E_{w_{\ell+1}} J^{\lambda_1}_{t-\ell-2}(\cdot)$ and $\delta E_{w_{\ell+2}} J^{\lambda_1}_{t-\ell-2}(\cdot)$ cancel and we have,

$$
\text{(DIFF.1)} = \delta^\ell (1-\delta) E_{\underline{w}} R(f^{\circ\ell}(x,\underline{v},\underline{w})) - \delta^\ell (1-\delta) \lambda_1 .
$$

Following the same sequence of reasoning, we get the expression for (DIFF.2) provided below:

$$(\text{DIFF.2}) = \delta^{\ell}(1-\delta)\lambda_2 - \delta^{\ell}(1-\delta)E_{\underline{w'}}R(f^{o\ell}(x,\underline{v},\underline{w'})).$$

Summing the expressions (DIFF.1) and (DIFF. 2),

$$\begin{aligned} DIFF &\leq (DIFF.1)+(DIFF.2) \\ &= \delta^{\ell}(1-\delta)\lambda_2 - \delta^{\ell}(1-\delta)\lambda_1 \\ &\quad +\delta^{\ell}(1-\delta)E_{\underline{w}}R(f^{o\ell}(x,\underline{v},\underline{w})) - \delta^{\ell}(1-\delta)E_{\underline{w'}}R(f^{o\ell}(x,\underline{v},\underline{w'})). \end{aligned}$$

The initial w_1 and w'_1 share the same distribution because it is dependent on the original state of the arm x at time t. Also, since $\underline{v}$ are identical, the expression involving the expectations cancel and we have that DIFF is bounded above by

$$\begin{aligned} \text{DIFF} \leq &\ \delta^{\ell}(1-\delta)\lambda_2 - \delta^{\ell}(1-\delta)\lambda_1 \\ = &\ (\lambda_2-\lambda_1)\delta^{\ell}(1-\delta) < 0, \quad \forall\delta < 1. \end{aligned}$$

■

We now present the result that multiarmed bandit problems with constant response delay are indexable.

Theorem I.1 *The multiarmed bandit problem with constant response delay ℓ is indexable.*

Proof. First, we have $\Delta J_t^{\lambda}(z)$ decreasing in λ, $\forall t,z$, and it is easy to see that $\Delta J_t^{0}(z) > 0$, and $\Delta J_t^{\infty}(z) < 0$. Thus, to show that a well-defined λ exists such that $\Delta J_t^{\lambda}(z) = 0$, it suffices to show that $\Delta J_t^{\lambda}(z)$ is continuous in λ. We do this by induction.

When $t = \ell$, we have $\Delta J_{\ell}^{\lambda}(x,v^1,\ldots,v^{\ell}) = E_w(R(x)-\lambda)$, and $J_{\ell}^{\lambda}(x,v^1,\ldots,v^{\ell}) = \max\{E_w(R(x)),\lambda\}$, which are clearly continuous in λ for all z. Suppose that $\Delta J_{t-1}^{\lambda}(z)$

and $J_{t-1}^{\lambda}(z)$ are continuous in λ for all z. Then,

$$\Delta J_t^{\lambda}(x,v^1,...,v^{\ell}) = \delta E_w\{J_{t-1}^{\lambda}(f(x,v^1,w),v^2,...,v^{\ell},1) - J_{t-1}^{\lambda}(f(x,v^1,w),v^2,...,v^{\ell},0)\}$$

and

$$J_t^{\lambda}(x,v^1,...,v^{\ell}) = \max\{E_w J_{t-1}^{\lambda}(f(x,v^1,w),v^2,...,v^{\ell},1), E_w J_{t-1}^{\lambda}(f(x,v^1,w),v^2,...,v^{\ell},0)\}$$

are clearly continuous in λ. Moreover, as $R(x)$ is uniformly bounded (by problem assumption), $J_t^{\lambda}(z)$ converges as $t \to \infty$. Hence, there is a well-defined λ such that $\Delta J^{\lambda}(z) = 0$. ■

3.2 Indexability of Stationary Random Delay

In many practical settings, delays may be random. We show that the indexability result can be generalized to bandit problems with stationary random delays, in which the delayed responses do not crossover (i.e., the stochastic delay $\ell \in \{m, m+1\}$ for some fixed integer m). If, however, the randomness in the delay lengths permits the delayed responses to crossover (i.e., $\ell \in \{m, \ldots m+K\}$, $K > 1$), then the bandit problem is no longer indexable.

Theorem I.2 *The bandit with stationary random delay is indexable if the delayed responses do not crossover. However, indexability need not hold if the delayed responses are allowed to crossover.*

Proof. See Appendix B.

This result is analogous to the inventory systems with stochastic leadtimes. In particular, if the random delay process does not have order crossovers, the base-stock policy is shown to be optimal (e.g. Kaplan 1970, Muharremoglu and Yang 2008).

However, Robinson et al. (2001) show that the base-stock policy is no longer optimal when the order are allowed to crossover.

4 Numerical Work

In this section, we examine the Beta-Bernoulli learning model where the prior distribution of the success probability p of the Bernoulli random variable is characterized by a Beta distribution with parameters (α, β). The state of an arm, corresponding to this parameter (α, β), is updated in a Bayesian manner: to $(\alpha+1, \beta)$ after observing a success or to $(\alpha, \beta+1)$ after observing a failure. In other words, a bandit in state (α, β) is statistically equivalent to one that began with its success probability p having an a priori distribution uniform on $[0,1]$, and which has now shown $\alpha - 1$ successes and $\beta - 1$ failures in $\alpha + \beta$ pulls.

We compute the indices for the multiarmed bandit model with constant delay ℓ. Then, using the indices we examine the performance of the resulting MPI policy against an upperbound, and compare it to those of other existing closed-form indices.

4.1 Index Computation

Compared to the classical multiarmed bandit problem (with no delay), the indices from Theorems 1 and 2 do not have an equivalent representation as an optimal stopping-time problem. Therefore, an approach to compute the indices based on this property, which Gittins (1989) calls the *direct approach*, is not available. Instead, we adopt the *calibration approach* which uses dynamic programming value iteration (see Gittins 1989 for further discussion of both approaches).

The indices for the bandit problem without delay using the Beta-Bernoulli learning model have been computed and tabulated in Gittins (1989). We extend this table by adding the indices for delays $\ell \in \{1,2,3,4,5\}$ and discount factors $\delta \in \{0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99\}$, and make it available online for public use (see the authors' website). The indices have been computed using the subsidy/implementation framework, and by Proposition 1, the index values under other reward accounting methods are the same up to a constant factor, which does not affect the actions suggested by the MPI-index policy.

4.2 Numerical Simulation

In this section, we examine the performance of the MPI policy for the Beta-Bernoulli learning model (DeGroot 1970) with constant delays. We compute a performance upperbound by solving a relaxed multiarmed bandit problem in which the constraint that does not allow more than N arms pulled per period is only required to hold on average (see the Appendix C for the upperbound formulation). We use this to gauge the suboptimality of the MPI policy. We then compare its performance with the myopic policy (MYO) that maximizes the single-period reward (see for instance, Aviv and Pazgal 2002), and the closed-form index policies developed by: Caro and Gallien (2007, denoted CG), Brezzi and Lai (2002, BL), and Ginebra and Clayton (1995, GC), in which the respective index formulas (shown in Table 1) have been modified as in Caro and Gallien (2007) to account for delays.[1]

The simulation and the upper bound optimization codes are written in Matlab, and are available from the authors upon request. Using a discount rate of $\delta = 0.95$, we

[1]The index-specific coefficients of the CG and GC formulas where obtained through least-squares using a small sample of exact MPI values.

Table I.1: Closed-form index formulas. Here B denotes the Beta prior with parameters (α, β) and a *delayed-adjusted* variance equal to $V[B] = \alpha\beta\Big((\alpha+\beta)^2(\alpha+\beta+\sum_{\tau=1}^{\ell} v^\tau+1)\Big)^{-1}$, where v^τ is the τ-th delayed action. Accordingly, Y denotes a Bernoulli random variable with success probability $\alpha(\alpha+\beta)^{-1}$. For index-specific coefficients and functions, refer to the original articles.

Name	Closed-form Index	Index-specific
Myopic (MYO)	$E[B]$	
Caro-Gallien (CG)	$E[B]+z_{\delta,\ell}\sqrt{V[B]}\left(\frac{V[B]}{V[Y]}\right)^{c_{\delta,\ell}}$	$z_{\delta,\ell}, c_{\delta,\ell}$
Brezzi-Lai (BL)	$E[B]+\sqrt{V[B]}\psi\left(\frac{V[B]}{\ln(\frac{1}{\delta})V[Y]}\right)$	$\psi(\cdot)$
Ginebra-Clayton (GC)	$E[B]+k_{\delta,\ell}\sqrt{V[B]}$	$k_{\delta,\ell}$

run a series of simulations for 5 delay periods $\ell \in \{1,2,3,4,5\}$ for T periods such that $\sum_{t=T+1}^{\infty}\delta^t < 10^{-6}$ to approximate infinite horizon. We let our initial prior to be the uniform distribution, corresponding to the Beta distribution with parameter $(\alpha,\beta) = (1,1)$, as it best represents the initial state of knowledge. We did an extensive simulation study and here we show the results for a few representative instances.

The simulation results where the decision maker pulls 4 arms out of a total of 32 arms, i.e. $(S,N) = (32,4)$, are shown in Table 2. The first observation is that MPI index policy is near optimal since the suboptimality gap is very small. In general it was less than 4% in all the simulations we ran, and in most cases it was actually less than 2%. The gap has a slight tendency to increase with the length of the delay ℓ. This could suggest that the MPI policy becomes slightly worse. However, it could also be that the upperbound deteriorates with longer delays.

We also note that all the delay-incorporated closed-form index policies perform

Table I.2: Suboptimality gap for the MPI policy and the closed-form benchmark policies. $(\alpha,\beta) = (1,1)$, $(S,N) = (32,4)$, $\delta = 0.95$.

ℓ	MYO (%)	CG (%)	BL (%)	GC (%)	MPI (%)	UpperBnd
1	7.96	0.74	0.69	0.53	**0.51**	60.30
2	8.80	1.43	1.66	1.31	**1.53**	59.65
3	8.98	2.26	2.72	2.71	**2.15**	59.01
4	7.23	2.61	2.81	2.80	**2.82**	58.38
5	6.34	3.56	3.51	3.66	**3.63**	57.80

very close to the MPI policy and that the differences are not statistically significant. We attribute the performance similarity to the fact that all the values of the modified closed-form indices provide good approximations of the MPIs. We do however find that the myopic policy performs significantly worse than all other policies. This is to be expected because the myopic policy ignores the delayed actions as well as the future benefits from learning.

Computing a large table of necessary MPI's often requires high level of computational complexity. Our finding suggests that, in such cases, one should adjust the existing closed-form indices and use the policy as a substitute for the MPI policy and attain comparable results.

Furthermore, we find that as the number of projects S and the number of allowable pulls N increase while maintaining a constant ratio N/S, the suboptimality gap of the MPI policy approaches zero. The suboptimality gaps for $(S,N) = (32,4)$, $(S,N) = (160,20)$, and $(S,N) = (320,40)$ are shown in Table 3. Whittle (1988) initially conjectured that the MPI index policy is asymptotically optimal. This was largely validated by Weber and Weiss (1990) for finite-state restless bandits. Our results support the conjecture for infinite-state bandits with response delay.

Table I.3: Asymptotic suboptimality gap. $(\alpha, \beta) = (1, 1)$, $\delta = 0.95$.

	(S,N):	(32,4)	(S,N):	(160, 20)	(S,N):	(320,40)
ℓ	MYO (%)	MPI (%)	MYO (%)	MPI (%)	MYO (%)	MPI (%)
1	7.96	**0.51**	8.65	**0.64**	8.78	**0.07**
2	8.80	**1.53**	7.64	**0.65**	7.72	**0.28**
3	8.98	**2.15**	4.75	**0.47**	4.62	**0.45**
4	7.23	**2.82**	3.97	**0.87**	3.74	**0.68**
5	6.34	**3.63**	4.75	**1.60**	4.77	**1.17**

5 Conclusion

In this paper, we prove the indexability of the multiarmed bandit problem with response delay, where the delays are of arbitrary length and are allowed to be stationary random as long as the delayed responses do no crossover. We show that, under stationarity assumption, the problem is not indexable if the order is allowed to crossover. The MPI policy performs near optimally, and the closed-form index policies when adjusted for delay represent good estimations of the MPI and perform well.

Further refinements of these policies are worth studying. For example, Kaplan (1970) formulates a stochastic lead time process in which the delays for each period are identically distributed but statistically dependent random variables so that orders do not crossover. It would be worthwhile to examine whether our results hold for nonstationary random delays. Another interesting variation is to make the pulls *irrevocable*. That is, once an arm stops being pulled, it can never be pulled again. This can be a desirable property from a practical standpoint and the results available for the classical bandit problem show a high performance that might extend to the case with response delays (see Farias and Madan 2008).

APPENDICES

A. Equivalence Relation

There are potentially four different ways of accounting for the Lagrange multiplier λ, which are given below and are summarized in the following table.

Lagrange Multiplier λ Accounted:	As Subsidy for Not Pulling	As Cost for Pulling
When Decision Made	$\widehat{J}_t^\lambda$	$\widehat{H}_t^\lambda$
When Decision Implemented	J_t^λ	H_t^λ

Table I.4: Four different representations.

$$\begin{aligned}
\widehat{J}_t^\lambda(x,v^1,...,v^\ell) = \quad & R(x)v^1+\max\{\delta E_w\widehat{J}_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,1),\\
& \lambda+\delta E_w\widehat{J}_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,0)\}.\\
J_t^\lambda(x,v^1,...,v^\ell) = \quad & R(x)v^1+\lambda(1-v^1)+\max\{\delta E_w J_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,1),\\
& \delta E_w J_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,0)\}.\\
\widehat{H}_t^\lambda(x,v^1,...,v^\ell) = \quad & R(x)v^1+\max\{-\lambda+\delta E_w\widehat{H}_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,1),\\
& \delta E_w\widehat{H}_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,0)\}.\\
H_t^\lambda(x,v^1,...,v^\ell) = \quad & (R(x)-\lambda)v^1+\max\{\delta E_w H_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,1),\\
& \delta E_w H_{t-1}^\lambda(f(x,v^1,w),v^2,...,v^\ell,0)\}.
\end{aligned}$$

Proof of Proposition 1. Through induction on t it can be shown that $\Delta J_t^\lambda(z) = \Delta H_t^\lambda(z) = \Delta\widehat{J}_t^{\delta^\ell\lambda}(z) = \Delta\widehat{H}_t^{\delta^\ell\lambda}(z)$ for every (augmented) state z, where the operator Δ denotes the difference between the expected profits under the *Pull* and *NotPull* decisions. Clearly, if the difference is decreasing in λ for any of the formulations, then it is also decreasing for the other formulations. The Proposition follows from this observation. ■

B. Indexability of Stationary Random Delay without Order Crossover

We first establish the following monotonicity result using the coupling argument as was done previously for Proposition 2.

Proposition B.1. For all state z, $\Delta J_t^{\lambda}(z)$ is monotonically decreasing in λ for stationary random delay ℓ if the orders do not cross over, i.e., $\ell \in \{m, m+1\}$.

Proof. We show that if $\ell \in \{m, m+1\}$ then the problem is indexable.

We show that for any $\lambda_1 > \lambda_2$, $\Delta J_t^{\lambda_1}(z) < \Delta J_t^{\lambda_2}(z)$ for all z, via induction. For simplicity of illustration, we will assume that $m = 0$, or in other words, that the delay is uncertain between no delay and a delay of period 1. The structure of the proof remains identical for $m > 0$.

Base Case: $t = 1$

In the final decision period $t = 1$, there will be zero delay with probability p_0, and a delay of one period with probability p_1. We have,

$$\begin{aligned}
J_1^{\lambda}(x,1) = &\quad R(x) + \max\{p_0 E_{w_1} R(f(x,1,w_1)) + p_1 0, p_0\lambda + p_1 0\}, \\
J_1^{\lambda}(x,0) = &\quad \lambda + \max\{p_0 R(x) + p_1 0, p_0\lambda + p_1 0\}, \\
J_1^{\lambda}(x,\emptyset) = &\quad 0 + \max\{p_0 R(x) + p_1 0, p_0\lambda + p_1 0\}, \text{ and}
\end{aligned}$$

$$\Delta J_1^\lambda(x,1) = \quad p_0\{E_{w_1}R(f(x,1,w_1)) - \lambda\},$$
$$\Delta J_1^\lambda(x,0) = \Delta J_1^\lambda(x,\emptyset) = \quad p_0\{R(x) - \lambda\}.$$

All are clearly decreasing in λ.

Induction Step:

Suppose that $\forall z$, $\lambda_1 > \lambda_2$, $\Delta J_{t-1}^{\lambda_1}(z) < \Delta J_{t-1}^{\lambda_2}(z)$. We will show that $\forall z$, $\lambda_1 > \lambda_2$, $\Delta J_t^{\lambda_1}(z) < \Delta J_t^{\lambda_2}(z)$.

Again, there will be no delay with probability p_0, and a delay of one period with probability p_1. We have,

$$\begin{aligned}
J_t^\lambda(x,1) = \quad & R(x) + \max\{p_0(E_{w_1}R(f(x,1,w_1)) + \delta E_{w_2}J_{t-1}^\lambda(f(f(x,1,w_1),1,w_2),\emptyset)) \\
& \qquad + p_1(\delta J_{t-1}^\lambda(f(x,1,w_1),1)), \\
& \quad p_0(\lambda + \delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),\emptyset)) \\
& \qquad + p_1(\delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),0))\}, \\
J_t^\lambda(x,0) = \quad & \lambda + \max\{p_0(R(x) + \delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),\emptyset)) + p_1(\delta J_{t-1}^\lambda(x,1)), \\
& \quad p_0(\lambda + \delta J_{t-1}^\lambda(x,\emptyset)) + p_1(\delta J_{t-1}^\lambda(x,0))\}, \\
J_t^\lambda(x,\emptyset) = \quad & 0 + \max\{p_0(R(x) + \delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),\emptyset)) + p_1(\delta J_{t-1}^\lambda(x,1)), \\
& \quad p_0(\lambda + \delta J_{t-1}^\lambda(x,\emptyset)) + p_1(\delta J_{t-1}^\lambda(x,0))\}, \quad \text{and} \\
\Delta J_t^\lambda(x,1) = \quad & p_0(E_{w_1}R(f(x,1,w_1)) - \lambda) + p_0(\delta E_{w_2}J_{t-1}^\lambda(f(f(x,1,w_1),1,w_2),\emptyset) \\
& \quad - \delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),\emptyset)) \\
& \quad + p_1(\delta J_{t-1}^\lambda(f(x,1,w_1),1) - \delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),0)), \\
\Delta J_t^\lambda(x,0) = \quad & \Delta J_t^\lambda(x,\emptyset) = p_0(R(x) - \lambda) + p_0(\delta E_{w_1}J_{t-1}^\lambda(f(x,1,w_1),\emptyset) - \delta J_{t-1}^\lambda(x,\emptyset)), \\
& \quad + p_1(\delta J_{t-1}^\lambda(x,1) - \delta J_{t-1}^\lambda(x,0)).
\end{aligned}$$

For simplicity, we will examine the indexability for state $z = (x,0)$, but the identical argument holds for other states. We have,

$$
\begin{aligned}
\text{DIFF} \quad & \equiv \Delta J_t^{\lambda_1}(x,0) - \Delta J_t^{\lambda_2}(x,0) \\
& = p_0 \times \{(\lambda_2 - \lambda_1) + \delta E_{w_1} J_{t-1}^{\lambda_1}(f(x,1,w_1),\emptyset) - \delta J_{t-1}^{\lambda_1}(x,\emptyset) \\
& \qquad - \delta E_{w_1'} J_{t-1}^{\lambda_2}(f(x,1,w_1'),\emptyset) + \delta J_{t-1}^{\lambda_2}(x,\emptyset)\} \\
& \quad + p_1 \times \{\delta J_{t-1}^{\lambda_1}(x,1) - \delta J_{t-1}^{\lambda_1}(x,0) - \delta J_{t-1}^{\lambda_2}(x,1) + \delta J_{t-1}^{\lambda_2}(x,0)\}.
\end{aligned}
$$

After rewriting the expression and rearranging the terms, we have

$$
\begin{aligned}
\text{DIFF} = \quad & p_0 \times \{(\lambda_2 - \lambda_1) + \delta E_{w_1} J_{t-1}^{\lambda_1}(f(x,1,w_1),\emptyset)^{(NotPull)} - \delta J_{t-1}^{\lambda_1}(x,\emptyset)^{(Pull)} \\
& \qquad - \delta E_{w_1'} J_{t-1}^{\lambda_2}(f(x,1,w_1'),\emptyset)^{(NotPull)} + \delta J_{t-1}^{\lambda_2}(x,\emptyset)^{(Pull)}\} \\
& + p_1 \times \{\delta J_{t-1}^{\lambda_1}(x,1)^{(NotPull)} - \delta J_{t-1}^{\lambda_1}(x,0)^{(Pull)} \\
& \qquad - \delta J_{t-1}^{\lambda_2}(x,1)^{(NotPull)} + \delta J_{t-1}^{\lambda_2}(x,0)^{(Pull)}\} \\
& + p_0 \times \{\delta E_{w_1}[\Delta J_{t-1}^{\lambda_1}(f(x,1,w_1),\emptyset)]^+ - \delta E_{w_1'}[\Delta J_{t-1}^{\lambda_2}(f(x,1,w_1'),\emptyset)]^+ \\
& \qquad - \delta[\Delta J_{t-1}^{\lambda_1}(x,\emptyset)]^- + \delta[\Delta J_{t-1}^{\lambda_2}(x,\emptyset)]^- \ \} \ \} \\
& + p_1 \times \{\delta[\Delta J_{t-1}^{\lambda_1}(x,1)]^+ - \delta[\Delta J_{t-1}^{\lambda_2}(x,\emptyset)]^+ \\
& \qquad - \delta[\Delta J_{t-1}^{\lambda_1}(x,0)]^- + \delta[\Delta J_{t-1}^{\lambda_2}(x,0)]^-\}.
\end{aligned}
$$

Eliminating the bottom two expressions, which are both non-positive by the induction assumption, we have

$$
\begin{aligned}
\text{DIFF} \leq\ & p_0 \times \{(\lambda_2-\lambda_1)+\delta E_{w_1} J^{\lambda_1}_{t-1}(f(x,1,w_1),\emptyset)^{(NotPull)} - \delta J^{\lambda_1}_{t-1}(x,\emptyset)^{(Pull)} \\
& \qquad -\delta E_{w_1'} J^{\lambda_2}_{t-1}(f(x,1,w_1'),\emptyset)^{(NotPull)} + \delta J^{\lambda_2}_{t-1}(x,\emptyset)^{(Pull)}\} \\
& +p_1 \times \{\delta J^{\lambda_1}_{t-1}(x,1)^{(NotPull)} - \delta J^{\lambda_1}_{t-1}(x,0)^{(Pull)} \\
& \qquad -\delta J^{\lambda_2}_{t-1}(x,1)^{(NotPull)} + \delta J^{\lambda_2}_{t-1}(x,0)^{(Pull)}\} \\
=\ & p_0 \times \{(\lambda_2-\lambda_1) \\
& \quad +p_0 \times \{\delta E_{w'}(\lambda_1 + \delta J^{\lambda_1}_{t-2}(f(x,1,w_1),\emptyset)) \\
& \qquad -\delta(R(x)+\delta E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),\emptyset)) \\
& \qquad -\delta E_{w_1'}(\lambda_2+\delta J^{\lambda_2}_{t-2}(f(x,1,w_1'),\emptyset)) \\
& \qquad +\delta(R(x)+\delta E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),\emptyset)) \\
& \quad +p_1 \times \{\delta^2 E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),0) - \delta^2 J^{\lambda_1}_{t-2}(x,1) \\
& \qquad -\delta^2 E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),0) + \delta^2 J^{\lambda_2}_{t-2}(x,1)\}\}\} \\
& +p_1 \times \{p_0 \times \{\delta(R(x)+\lambda_1+\delta E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),\emptyset)) \\
& \qquad -\delta(\lambda_1+R(x)+E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),\emptyset)) \\
& \qquad -\delta(R(x)+\lambda_2+\delta E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),\emptyset)) \\
& \qquad +\delta(\lambda_2+R(x)+E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),\emptyset)) \\
& \quad +p_1 \times \{\delta(R(x)+\delta E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),0)) - \delta(\lambda_1+\delta J^{\lambda_1}_{t-2}(x,1)) \\
& \qquad -\delta(R(x)+\delta E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),0)) + \delta(\lambda_2+\delta J^{\lambda_2}_{t-2}(x,1))\}\}\} \\
=\ & p_0(\lambda_2-\lambda_1)+\delta(\lambda_1-\lambda_2)(p_0^2-p_1^2) \\
& +p_1\{\delta^2 E_{w_1} J^{\lambda_1}_{t-2}(f(x,1,w_1),0) - \delta^2 J^{\lambda_1}_{t-2}(x,1) \\
& \quad -\delta^2 E_{w_1'} J^{\lambda_2}_{t-2}(f(x,1,w_1'),0) + \delta^2 J^{\lambda_2}_{t-2}(x,1)\}.
\end{aligned}
$$

We now introduce the coupling argument. Consider the bandit with subsidy λ_1 starting from two different states. The first, which we refer to System A, starts from the augmented state $(f(x,1,w_1),0)$ at time $t-2$. The second, which we refer to as System B, starts from state $(x,1)$, but implements the same decisions as System A in the first stage, and after that, it follows its own optimal policy. Let π^* denote the optimal policy of System A.

Let $G^{\lambda_1}_{\pi^*_{t-2}}(z)$ represent the value of being in state z for λ_1 at time $t-2$ and following the policy π^*. Then we have,

$$J^{\lambda_1}_{t-2}(f(x,1,w_1),0) = G^{\lambda_1}_{\pi^*_{t-2}}(f(x,1,w_1),0), \quad \text{and} \quad J^{\lambda_1}_{t-2}(x,1) \geq G^{\lambda_1}_{\pi^*_{t-2}}(x,1).$$

The same coupling argument can be used for the bandit with subsidy λ_2, only that System A would start in state $(x,1)$ and System B would start in $(f(x,1,w'_1),0)$. Denoting π^{**} as the optimal policy of System A, we have

$$J^{\lambda_2}_{t-2}(x,1) = G^{\lambda_2}_{\pi^{**}_{t-2}}(x,1), \quad \text{and} \quad J^{\lambda_2}_{t-2}(f(x,1,w'_1),0) \geq G^{\lambda_2}_{\pi^{**}_{t-2}}(f(x,1,w'_1),0).$$

By subtracting these smaller values of $G^{\lambda_1}_{\pi^*_{t-2}}$ and $G^{\lambda_2}_{\pi^{**}_{t-2}}$, the right hand side of the inequality, and therefore DIFF, is bounded above by

$$\begin{aligned}\text{DIFF} \leq \;& p_0(\lambda_2-\lambda_1)+\delta(\lambda_1-\lambda_2)(p_0^2-p_1^2)\\ &+p_1\{\delta^2 E_{w_1} G^{\lambda_1}_{\pi^*_{t-2}}(f(x,1,w_1),0) - \delta^2 G^{\lambda_1}_{\pi^*_{t-2}}(x,1) && \text{(DIFF.1)}\\ &\quad +\delta^2 G^{\lambda_2}_{\pi^{**}_{t-2}}(x,1) - \delta^2 E_{w'_1} G^{\lambda_2}_{\pi^{**}_{t-2}}(f(x,1,w'_1),0)\}. && \text{(DIFF.2)}\end{aligned}$$

We now elaborate the expressions (DIFF.1) and (DIFF.2).

$$\begin{aligned}\text{(DIFF.1)} \;&= p_1\{\delta^2 E_{w_1}\{p_0(\lambda_1+R(f(x,1,w_1))u^*+\lambda_1(1-u^*)\\ &\qquad +\delta E_{w_2} G^{\lambda_1}_{\pi^*_{t-3}}(f(f(x,1,w_1),u^*,w_2),\emptyset))\\ &\qquad +p_1(\lambda_1+\delta E_{w_1} G^{\lambda_1}_{\pi^*_{t-3}}(f(x,1,w_1),u^*))\}\\ &\quad -\delta^2\{p_0(R(x)+E_{w_1}R(f(x,1,w_1))u^*+\lambda_1(1-u^*)\\ &\qquad +\delta E_{w_2} G^{\lambda_1}_{\pi^*_{t-3}}(f(f(x,1,w_1),u^*,w_2),\emptyset))\\ &\qquad +p_1(R(x)+E_{w_1} G^{\lambda_1}_{\pi^*_{t-3}}(f(x,1,w_1),u^*))\}\}\\ &= p_1\{\delta^2(\lambda_1-R(x))\}.\end{aligned}$$

Following the same argument, we have

(DIFF.2) $= p_1\{\delta^2(R(x)-\lambda_2)\}$.

Thus, after summing the expressions, we have

$$\begin{aligned}
\text{DIFF} = & \; p_0(\lambda_2-\lambda_1)+\delta(\lambda_1-\lambda_2)(p_0^2-p_1^2)+p_1\delta^2(\lambda_1-\lambda_2) \\
= & \; (\lambda_2-\lambda_1)(p_0-\delta p_0^2+\delta p_1^2-\delta^2 p_1) \\
= & \; (\lambda_2-\lambda_1)(p_0(1-\delta p_0)+\delta p_1(p_1-\delta)) < 0.
\end{aligned}$$

Notice that if $p_0 = 1$ & $p_1 = 0$ or $p_0 = 0$ & $p_1 = 1$, the above inequality reduces to, respectively,

$$(\lambda_2-\lambda_1)(1-\delta) < 0, \quad \text{and} \quad (\lambda_2-\lambda_1)\delta(1-\delta) < 0,$$

which is consistent with the result of Proposition 2. ■

Proposition B.2. If the delayed responses are allowed to crossover, then $\Delta J_t^\lambda(z)$ is not necessarily monotonically decreasing.

Proof. We provide an example of a range of λ's in which $\Delta J_t^\lambda(z)$ is increasing when the delayed responses are allowed to crossover. In particular, consider $\ell \in \{0,2\}$, $z = (x,0,\emptyset)$ at time $t = 4$, and $J_0^\lambda(\cdot) = 0$. We have,

$$\begin{aligned}
J_4^\lambda(x,0,\emptyset) &= \lambda+\max\{\delta J_3^\lambda(x,\emptyset,1),\delta J_3^\lambda(x,\emptyset,0)\} \\
\Delta J_4^\lambda(x,0,\emptyset) &= \delta\{J_3^\lambda(x,\emptyset,1)-J_3^\lambda(x,\emptyset,0)\}.
\end{aligned}$$

We elaborate the necessary $J_3^\lambda(\cdot)$'s and $J_2(\cdot)'s$,

$$
\begin{aligned}
J_3^\lambda(x,\emptyset,1) = &\quad \max\{R(x)+\delta E_w J_2^\lambda(f(x,1,w),1,\emptyset),\lambda+\delta J_2^\lambda(x,1,\emptyset)\}, \\
J_3^\lambda(x,\emptyset,0) = &\quad \max\{R(x)+\delta E_w J_2^\lambda(f(x,1,w),0,\emptyset),\lambda+\delta J_2^\lambda(x,0,\emptyset)\}, \\
\delta E_w J_2^\lambda(f(x,1,w),1,\emptyset) = &\quad \delta E_w R(f(x,1,w)) \\
&\qquad +\delta^2 E_w E_{w'} \max\{R(f(f(x,1,w),1,w')),\lambda\}, \\
\delta J_2^\lambda(x,1,\emptyset) = &\quad \delta R(x)+\delta^2 E_w \max\{R(f(x,1,w)),\lambda\}, \\
\delta E_w J_2^\lambda(f(x,1,w),0,\emptyset) = &\quad \delta\lambda+\delta^2 E_w \max\{R(f(x,1,w)),\lambda\}, \\
\delta J_2^\lambda(x,0,\emptyset) = &\quad \delta\lambda+\delta^2 \max\{R(x),\lambda\}.
\end{aligned}
$$

We consider the following independent binary random process as shown in the figure.

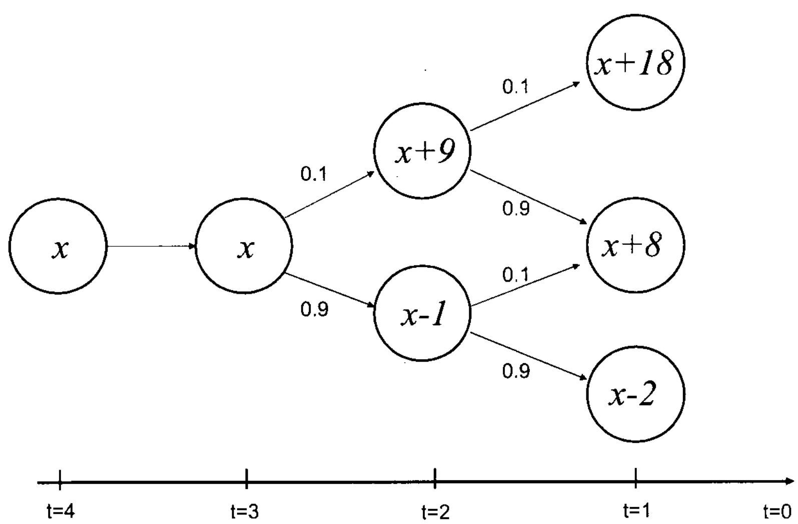

Figure I.1: Independent binary random process.

Let $R(x) = x$, and for simplicity let us take $x = 1$, and let $\lambda_1 = 1.67$, and $\lambda_2 = 1.55$, with $\delta = 0.9$. Substituting these values into the expression above, we have

$J_3^{\lambda_1}(x,\emptyset,1) = 4.60, \quad J_3^{\lambda_1}(x,\emptyset,0) = 4.53, \quad J_3^{\lambda_2}(x,\emptyset,1) = 4.39, \quad J_3^{\lambda_2}(x,\emptyset,0) = 4.34,$

giving us,

$$\Delta J_4^{\lambda_1}(x,0,\emptyset) \quad = \delta\{J_3^{\lambda_1}(x,\emptyset,1) - J_3^{\lambda_1}(x,\emptyset,0)\} \quad = 0.9(0.067) = 0.0603$$
$$\Delta J_4^{\lambda_2}(x,0,\emptyset) \quad = \delta\{J_3^{\lambda_2}(x,\emptyset,1) - J_3^{\lambda_2}(x,\emptyset,0)\} \quad = 0.9(0.055) = 0.0495.$$

Or in other words, although $\lambda_1 > \lambda_2$, we have $\Delta J_4^{\lambda_1}(x,0,\emptyset) > \Delta J_4^{\lambda_2}(x,0,\emptyset)$, which implies that $\Delta J_4^{\lambda}(x,0,\emptyset)$ is increasing in this interval. ■

Proof of Theorem 2. The Theorem is clear by following the outline of the proof of Theorem 1, and using the results of Proposition B.1 and B.2. ■

C. Upperbound Formulation

So far, we have focused on formulating a heuristic because the dynamic programming formulation that defines the optimal policy is intractable. In this section, we formulate a tractable Lagrangian upperbound of the problem by decoupling the dynamic program into S independent arms. The upperbound enables us to provide a suboptimality guarantee of the resulting index policy.

Proposition C.1. Define the following function:

$$\begin{aligned} L_t^{\lambda}(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^{\ell}) = \quad & N\lambda \\ & + \max_{\mathbf{u}}\{\textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 + \delta \mathrm{E}_{\mathbf{w}} L_{t-1}^{\lambda}(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^{\ell}, \mathbf{u})\} \\ & \quad \text{s.t.} \quad \mathbf{u} \in \{0,1\}^S, \end{aligned}$$

for $t > \ell$, and for $t \leq \ell$,

$$\begin{aligned} L_t^{\lambda}(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^{\ell}) = \quad & N\lambda + \textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 + \delta \mathrm{E}_{\mathbf{w}} L_{t-1}^{\lambda}(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^{\ell})\} \\ L_0^{\lambda}(\cdot) = \quad & 0, \end{aligned}$$

where the λ represents the cost when the arm is *actually pulled*, and

$$L_t^*(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell) = \quad \min_\lambda L_t^\lambda(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell).$$

Then,

$$J_t^*(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell) \leq L_t^*(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell) \leq L_t^\lambda(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell).$$

Proof. We prove by induction. For $t \leq l$, it is clear that the following holds given that $\sum_{s=1}^S v^t \leq N, \quad \forall t \leq \ell$.

$$\begin{aligned} J_1^*(\mathbf{x}, \mathbf{v}^1) &= \textstyle\sum_{s=1}^S R_s(x_s) v_s^1 \\ &\leq N\lambda + \textstyle\sum_{s=1}^S (R_s(x_s) - \lambda) v_s^1 \\ &= L_1^\lambda(\mathbf{x}, \mathbf{v}^1), \end{aligned}$$

and

$$\begin{aligned} J_t^*(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^t) &= \textstyle\sum_{s=1}^S R_s(x_s) v_s^1 + \delta \mathrm{E}_\mathbf{w} J_{t-1}^*(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^t) \\ &\leq N\lambda + \textstyle\sum_{s=1}^S (R_s(x_s) - \lambda) v_s^1 + \delta \mathrm{E}_\mathbf{w} L_{t-1}^\lambda(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^t) \\ &= L_t^\lambda(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^t). \end{aligned}$$

For $t \geq \ell + 1$, suppose $J_{t-1}^* < H_{t-1}^\lambda$. Then,

$$\begin{aligned} J_t^*(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell) = \quad & \max_{\mathbf{u} \in \{0,1\}^S : \sum_{s=1}^S u_s \leq N} \Big\{ \textstyle\sum_{s=1}^S R_s(x_s) v_s^1 \\ & \qquad + \delta \mathrm{E}_\mathbf{w} J_{t-1}^*(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^\ell, \mathbf{u}) \Big\} \\ \leq \quad & \max_{\mathbf{u} \in \{0,1\}^S : \sum_{s=1}^S u_s \leq N} \Big\{ N\lambda + \textstyle\sum_{s=1}^S (R_s(x_s) - \lambda) v_s^1 \\ & \qquad + \delta \mathrm{E}_\mathbf{w} J_{t-1}^*(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^\ell, \mathbf{u}) \Big\} \\ \leq \quad & \max_{\mathbf{u} \in \{0,1\}^S : \sum_{s=1}^S u_s \leq N} \Big\{ N\lambda + \textstyle\sum_{s=1}^S (R_s(x_s) - \lambda) v_s^1 \\ & \qquad + \delta \mathrm{E}_\mathbf{w} L_{t-1}^\lambda(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^\ell, \mathbf{u}) \Big\} \\ \leq \quad & \max_{\mathbf{u} \in \{0,1\}^S} \Big\{ N\lambda + \textstyle\sum_{s=1}^S (R_s(x_s) - \lambda) v_s^1 \\ & \qquad + \delta \mathrm{E}_\mathbf{w} L_{t-1}^\lambda(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^\ell, \mathbf{u}) \Big\} \\ = \quad & L_t^\lambda(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^\ell). \end{aligned}$$

The first inequality follows because $\sum_{s=1}^{S} v_s^1 \leq N$, and the second inequality holds because of the induction assumption. The final inequality is because it is an optimization problem defined over a larger set. ■

We next show that the above expression for $L_t^\lambda(\mathbf{x}, \mathbf{v}^1, ..., \mathbf{v}^\ell)$ can be formulated more simply in terms of single-arm problems. Such similar decomposition has been shown previously without delay (see Caro and Gallien 2007, Bertsimas and Mersereau 2007).

Proposition C.2.

i)For $t \leq \ell$, $L_t^\lambda(\mathbf{x}, \mathbf{v}^1, ..., \mathbf{v}^t) = N\lambda \sum_{\tau=1}^{t} \delta^{\tau-1} + \sum_{s=1}^{S} L_{t,s}^\lambda(x_s, v_s^1, v_s^2, ..., v_s^t)$, where

$$L_{t,s}^\lambda(x_s, v_s^1, ..., v_s^t) = \quad (R(x_s) - \lambda)v_s^1 + \delta E_w L_{t-1,s}^\lambda(f(x, v^1, w), v^2, ..., v^t).$$

ii) For $t \geq \ell + 1$,

$$L_t^\lambda(\mathbf{x}, \mathbf{v}^1, ..., \mathbf{v}^t) = N\lambda \sum_{\tau=1}^{t} \delta^{\tau-1} + \sum_{s=1}^{S} L_{t,s}^\lambda(x_s, v_s^1, ..., v_s^t),$$

where

$$\begin{aligned} L_{t,s}^\lambda(x_s, v_s^1, ..., v_s^t) = \quad & \max\{(R(x_s) - \lambda)v_s^1 + \delta E_w L_{t-1,s}^\lambda(f(x, v^1, w), v^2, ..., v^t, 1), \\ & (R(x_s) - \lambda)v_s^1 + \delta E_w L_{t-1,s}^\lambda(f(x, v^1, w), v^2, ..., v^t, 0)\}. \end{aligned}$$

Proof. We prove by induction.

i) For delay of ℓ, $t = \ell + 1$ is where we'll make the final decision. Hence, $L_\ell^\lambda(\mathbf{x}, \mathbf{v}^1, ..., \mathbf{v}^\ell)$ can be considered a constant where the decisions $(\mathbf{v}^1, ..., \mathbf{v}^\ell)$ gets carried out. We first evaluate the quantity for $t \leq \ell$. $L_0^\lambda(\cdot) = 0$. First by Proposition C.1, we have:

$$\begin{aligned} L_1^\lambda(\mathbf{x}, \mathbf{v}^1) \quad &= N\lambda + \textstyle\sum_{s=1}^{S} (R_s(x_s) - \lambda)v_s^1 + \delta E_w\{L_0^\lambda(\cdot)\} \\ &= N\lambda + \textstyle\sum_{s=1}^{S} L_{1,s}^\lambda(x_s, v_s^1). \end{aligned}$$

Then, if we assume the expression holds for L_{t-1}^λ, we have

$$
\begin{aligned}
L_t^{\lambda}(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^t) &= N\lambda + \textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 + \delta E_{\mathbf{w}}\{L_{t-1}^{\lambda}(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2)\} \\
&= N\lambda + \textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 \\
&\quad +\delta E_{\mathbf{w}}\{N\lambda \textstyle\sum_{\tau=1}^{t-1}\delta^{\tau-1} + \textstyle\sum_{s=1}^{S} L_{t-1,s}^{\lambda}(f(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^t)\} \\
&= N\lambda \textstyle\sum_{\tau=1}^{t}\delta^{\tau-1} + \textstyle\sum_{s=1}^{S}\{(R_s(x_s) - \lambda)v_s^1 \\
&\quad +\delta E_{w_s} L_{t-1,s}^{\lambda}(f(x_s, v_s^1, w_s), v_s^2, \ldots, v_s^t)\} \\
&= N\lambda \textstyle\sum_{\tau=1}^{t}\delta^{\tau-1} + \textstyle\sum_{s=1}^{S} L_{t,s}^{\lambda}(x_s, v_s^1, \ldots, v_s^t).
\end{aligned}
$$

ii) Now suppose $t \geq \ell + 1$, and the expression holds for $t-1$. Then, again from Proposition C.1, we have the following expression:

$$
\begin{aligned}
L_t^{\lambda}(\mathbf{x}, \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}_s^t) &= N\lambda + \max_{\mathbf{u}}\{\textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 \\
&\quad +\delta E_{\mathbf{w}} L_{t-1}^{\lambda}(f(\mathbf{x}, \mathbf{v}^1, \mathbf{w}), \mathbf{v}^2, \ldots, \mathbf{v}^l, \mathbf{u})\} \\
&= N\lambda + \max_{\mathbf{u}}\{\textstyle\sum_{s=1}^{S}(R_s(x_s) - \lambda)v_s^1 + \delta E_{\mathbf{w}}\{N\lambda \textstyle\sum_{\tau=1}^{t-1}\delta^{\tau-1} \\
&\quad + \textstyle\sum_{s=1}^{S} H_{t-1,s}^{\lambda}(f(x_s, v_s^1, w), v_s^2, \ldots, v_s^t, u_s)\}\} \\
&= N\lambda \textstyle\sum_{\tau=1}^{t}\delta^{\tau-1} + \textstyle\sum_{s=1}^{S} \max_{\mathbf{u}}\{(R_s(x_s) - \lambda)v_s^1 \\
&\quad +\delta E_{\mathbf{w}} L_{t-1,s}^{\lambda}(f(x_s, v_s^1, w), v_s^2, \ldots, v_s^t)\} \\
&= N\lambda \textstyle\sum_{\tau=1}^{t}\delta^{\tau-1} + \textstyle\sum_{s=1}^{S} L_{t,s}^{\lambda}(x_s, v_s^1, v_s^2, \ldots, v_s^t).
\end{aligned}
$$

■

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CHAPTER II

Optimal Time Allocation Policy for Entrepreneurial Process Improvement

1 Introduction

Consider the following vignette (based on a blend of several true stories) of a time-constrained entrepreneur seeking revenue growth:

> Susan runs a small marketing agency that produces advertising materials to support her clients' needs. Her current list of services appeals to a small but loyal client base, from whom she can always generate extra revenue by working harder and bidding for more orders. Susan wants to grow her firm by offering higher margin services to a more high-end client base, but doing so would require her to spend time to learn about these new services and how to market them. At the same time, she feels that she spends too much of her time fighting fires, such as customer complaints or urgent questions from employees. She wants to reduce the frequency of all such disruptions, but that would also require her to spend time. Unable to do everything at once, Susan feels that there is not enough time in a day, and that her lack of time is what prevents her from growing her business.

In this paper, we develop a time-management framework for entrepreneurial process improvement. We classify an entrepreneur's daily activities into four categories, characterized by their impact on the current and future revenue rate and available time (defined more precisely later): revenue generation (earning money from existing business), revenue enhancement (developing new business, e.g. through product development or market research), fire-fighting (dealing with various crises), and process improvement (reducing the frequency of such crises). We outline a stylized time allocation model and present the optimal policy which prescribes how entrepreneurs like Susan should prioritize between the four activity categories.

Time management, due to its general appeal, has long been a major theme in the popular press. In "The Effective Executive," Drucker (1967) asserts that time is the most important resource that any executive can manage. Mackenzie (1997) warns that managers often wrongly associate "being busy" with "being productive," and identifies common "*time traps*" that they should avoid (e.g. uninvited visitors, phone interruptions, unnecessary paperwork or meetings). Covey (1989) classifies activities as a function of their urgency and importance, and claims that more time should be spent on activities that are important but not necessarily urgent. The central theme of all self-help books is that when one recognizes time as a resource and manages it well, more goals can often be achieved in less time.

While time management is important for all executives, two factors make it particularly relevant for entrepreneurs. First, an entrepreneur's time is, almost without exception, the bottleneck resource in their organization (e.g. Evans et al. 2004, Flamholtz 1986, Perlow 1999); this is also confirmed by hundreds of for-profit and social entrepreneurs participating in various programs at the major university in which the authors conduct research, as well as by the series of case studies the authors are involved in. In particular, although cash is sometimes thought of as being the most constrained resource (e.g. Archibald et al. (2002)), most entrepreneurs agree that, if they had more time, they could raise more capital from investors or donors, or earn more cash by selling more services to more customers. From that perspective, how entrepreneurs decide to use their scarce time is perhaps the most important resource allocation decision they make. Second, in contrast to most managers, entrepreneurs have (almost) complete control over how they use their time. In short, entrepreneurs are more able to manage their own time than most others, and how they use their time has a more direct effect on the firm's performance than is the case in larger firms.

We find that entrepreneurs should first invest time in process improvement until the process reliability reaches a certain threshold, then in revenue enhancement until the revenue rate reaches a certain threshold, and only then spend time generating revenue. In particular, entrepreneurs with lower initial revenue rates should invest more time in process improvement and in revenue enhancement, ultimately earning revenue at a higher rate than if they were endowed with a higher initial revenue rate. Our model formally links time with money and introduces a framework for evaluating the opportunity cost of an entrepreneur's time.

The rest of the paper is structured as follows. In §2, we review the related literature. In §3, we discuss the entrepreneurial setting and motivate our modeling assumptions. In §4, we present the dynamic programming (DP) framework for time allocation. We characterize the entrepreneur's optimal time allocation policy and discuss the opportunity cost of time in §5, and in §6 we extend the model to accommodate stochastic process deterioration and uncertainty in revenue enhancement and provide numerical illustrations of the optimal policy. In §7, we highlight the performance difference between the optimal policy and two commonly employed (well-intentioned) time allocation behaviors within the context of the model, and illustrate why our intuition can be misleading. We conclude in §8. All proofs appear in the online appendix.

2 Literature Review

Our work contributes to the operations management (OM) literature on entrepreneurial operations management and on process improvement, and to the economics literature on time allocation and on intertemporal decision making.

The challenges faced by entrepreneurs have received relatively little attention from the operations management community. Sommer and Loch (2004) offers strategic insights for coping with risk and complexity for companies that must adapt and innovate in dynamic environments. On a more operational level, Archibald et al. (2002), assuming that start-up companies maximize long-term survival probability rather than profit, compare inventory decisions between an established firm and a start-up firm. Under a similar assumption, Swinney et al. (2005) examine the effect of competition on startup's capacity decisions. Babich and Sobel (2004) link start-up companies' operational decisions with financial decisions and examine the optimal timing for an initial public offering (IPO), whereas Joglekar and Levesque (2009) examine how to allocate capital between improving product quality and marketing efforts to maximize firm valuation. In contrast to these studies, which characterize entrepreneurs by their cash constraint, we consider entrepreneurs' time as the main bottleneck resource of entrepreneurial companies, as is the case when they are in their growth phase.

Furthermore, our model presents a framework for process improvement. Fine and Porteus (1989) study the economics of gradual monetary investments in process improvement to save on costs in the future using a Markov decision process model, and Bernstein and Kok (2009) studies a similar problem in an assembly network setting. Using similar models, Fine (1988), Li and Rajagopalan (1998) and Dada and Marcellus (1994) study the impact of learning, which reduces the probability of future process deterioration, and Gong et al. (1997) study the tradeoff between a permanent fix and a partial fix. We contribute to the process improvement literature by focusing on investing time rather than money for improving operations and by considering revenue enhancement as well as process improvement.

In the economics literature, studies of time allocation abound, usually focusing on the decision where an agent seeks to optimally balance their time between work and leisure (e.g. Becker 1965, Baucells and Sarin 2007). In contrast, the time management decision problem where an agent allocates her time to various work-related tasks to maximize a work-related objective, has received less attention. Radner and Rothschild (1975) examine the properties of three intuitive heuristics for allocating effort for a manager who must simultaneously handle multiple activities. Seshadri and Shapira (2001) build on this framework by examining the feasibility of achieving a given long-term goal under various heuristics managers commonly employ. While these models give insights on how to manage time in order to meet an objective, they assume that the available time is exogenously given. Our work contributes to the time allocation literature by explicitly recognizing that the time available in the future can be increased through process improvement activities today.

The traditional approach to intertemporal trade-offs regarding money is the discounted utility (DU) framework initially proposed by Samuelson (1937). Despite the inconsistencies inherent in the DU framework as a model for intertemporal decision making (Fredrick et al. 2002), it nonetheless remains the norm for intertemporal financial valuation (Copeland et al. 2005). This is due to the existence of an interest rate, $r > 0$, set by market forces, which effectively makes one dollar tomorrow worth $\frac{1}{1+r} = \delta$ dollar today. Many psychologists however have found that people inherently perceive time differently from money (e.g. Soman 2001, LeClerc et al. 1995, Devoe and Pfeffer 2007, Okada and Hoch 2004), and in particular, Zauberman and Lynch (2005) show that people discount future time more heavily than future money. Nonetheless, no formal framework describing intertemporal trade-offs regarding time exists. We contribute to the discounting literature by presenting one possible normative framework for how people *should* discount time.

3 Entrepreneurial Operations

Entrepreneurship[1] is a vital component of the economy, employing over half of all private sector employees, and having generated 64 percent of new jobs annually over the last 15 years (U.S. Census Bureau 2009). Entrepreneurial activities fuel innovations in products and services covering a wide range of industries (Shane and Ulrich 2004), including technology startups, creative marketing, nonprofit organizations, petroleum distribution, legal services, and senior care. In this section, we present and discuss our assumptions, that (i) entrepreneurs proactively create demand, (ii) entrepreneurs are responsible for all functions within the firm but cannot easily delegate, and (iii) entrepreneurs' daily activities can be classified into four categories.

3.1 Endogenous Demand Creation

Entrepreneurs are often credited as being the agents in society who endogenously *create* demand. For example, Schumpeter (1934) argues that while changes in consumer demand can sometimes drive innovation, the reverse case where innovation creates consumer demand is more often true. Hayek (1945) takes the view that economic de-

[1]We refer to entrepreneurial firms as the small businesses defined by the U.S. Small Business Administration (SBA), i.e., independent business having fewer than 500 employees. This paper focuses more on entrepreneurial, i.e., growth-oriented firms than on established small businesses.

velopment is due to the "constant small changes" caused by entrepreneurs who use their "knowledge of particular circumstances of time and place" to arbitrage disjoint markets. Similarly, Kirzner (1997) asserts that entrepreneurs generate profit opportunities by discovering earlier errors and argues that "whereas each neoclassical decision-maker operates in a world of *given* price and output data, the entrepreneur operates to *change* price/output data."

We mirror the proactive nature of the entrepreneur's demand generating process by assuming that entrepreneurs are capable of generating extra demand for existing products and services or discovering new revenue opportunities when they allocate time to those activities.

3.2 Lack of Delegation Capacity

Much of an entrepreneur's activity relies on tacit knowledge, or "sticky information" (von Hippel 1994), which makes many entrepreneurial tasks difficult to delegate. Moreover, the organizational structure of an entrepreneurial firm, as opposed to professionally managed established firms, requires the entrepreneur to be involved in all functions. Flamholtz (1986) observes that "an entrepreneurial firm has an informal organizational structure with overlapping and undefined responsibilities . . . a CEO who knows everything that is going on and pays attention to the smallest details . . . [is] beneficial and necessary for the company."

Although we recognize the importance team structures have on innovation quality (Girotra et al. 2009), we will abstract away from the structural level of an organization and accordingly model the entrepreneur as the sole processor of the firm, unable to delegate material authorities to others, making her time allocation critical to the firm's

	Short-term	Long-term
Revenue-focused	RG	RE
Process-focused	FF	PI

Table II.1: Activity Classification

profitability.

3.3 Classification of Activities

We formally classify the activities that the entrepreneur can engage in at any point in time into the following four categories, which are further illustrated in Table II.1 based on their revenue/process dimension and short-term/long-term impact.

(RG): "Revenue Generation,"
or spending time to earn revenue at the prevailing rate,

(RE): "Revenue Enhancement,"
or investing time to increase the revenue rate,

(FF): "Fire Fighting,"
or spending time to attend to random urgent disruptions, and

(PI): "Process Improvement,"
or investing time to increase the process reliability.

Revenue generation refers to short-term activities that generate extra income for this period but do not permanently improve the revenue stream (rate). Revenue enhancement refers to growth activities such as creating new products or businesses to permanently improve the entrepreneurs' revenue rate. Fire fighting (Radner and Rothschild 1975) refers to urgent activities that must be attended to immediately but do not contribute to current or future revenue rate or available time (e.g. short term fixes). Finally, process improvement (Fine and Porteus 1989) refers to the activities of a preventive maintenance nature that reduce the need for fire-fighting and hence save time in the future. In practice, activities do sometimes fall in multiple categories, but treating this classification as being strict helps to bring out our main insights more strongly.

4 Model

In this section, we introduce the basic model and the DP framework. We then present the notions of time discounting and return on time invested (ROTI), and discuss our assumptions.

4.1 Dynamic Programming Framework

We characterize the entrepreneur's time allocation decision using an infinite horizon discounted DP model. Each time period represents a short time interval (e.g. one day or a half day), representing the duration of a typical activity. We assume the length of an entrepreneurial growth phase to be several months to a few years, under which approximating the entrepreneur's time horizon as infinite is a mild assumption.

Our model focuses on entrepreneurial firms in their growth phase, during which cash is not the key constraint. In particular, our model applies to two classes of entrepreneurs. The first class of entrepreneurs are well-funded by donors or investors who provide them with the capital necessary for growth as well as a constant wage B each period. The second class of entrepreneurs are those who can access the capital necessary for growth by leveraging their established track records and reputations (e.g.

good relationship with local banks, equity sharing agreements with various partners). These entrepreneurs have an established constant net base profit (after operating costs) of $B > 0$ per period (e.g. from standing orders, loyal customers) and in each period, they spend a fixed (small) portion of time serving this loyal customer base. Thus, our model discusses how to spend the *remaining* time for growth.

In each time period, the entrepreneur dedicates her time to a single activity. At the start of each period, a crisis may erupt, in which case the entrepreneur must dedicate the period to fire-fighting (FF). However, if there is no crisis, she may invest that period's time in productive activities such as revenue generation (RG), revenue enhancement (RE), or process improvement (PI). We let $a^t \in \{FF, RG, RE, PI\}$ denote the action of allocating time period t to one of these activities.

The state of the entrepreneur in each time period can be described by (i) whether or not there is a crisis, (ii) the current revenue rate (if she were to generate revenue), and (iii) the process reliability for future periods. This can be captured by the 3-dimensional state variable, $S^t = (C^t, R^t, q^t)$, described next.

- $C^t \in \{0, 1\}$ indicates the availability of time in period t. If $C^t = 0$, then time is not available in period t because the entrepreneur must "put out fires" (FF). If $C^t = 1$, then time is available for the entrepreneur to engage in a productive activity.

- $R^t \in \{R_m \mid 0 < R_0 < \ldots < R_m < R_{m+1} < \ldots < R_M\}$ refers to the revenue rate of the entrepreneur in period t, if she decides to spend that period generating revenue (RG). If she spends the time period on revenue enhancement (RE), the revenue rate would increase from R_m to R_{m+1}. A higher index denotes a better state.

Current State S^t	Decision	Immediate Reward	Future State S^{t+1}	Probability
$(0, R_m, q_n)$	FF	B	$(1, R_m, q_n)$	q_n
	(default)		$(0, R_m, q_n)$	$1 - q_n$
$(1, R_m, q_n)$	RG	$R_m + B$	$(1, R_m, q_n)$	q_n
			$(0, R_m, q_n)$	$1 - q_n$
	RE	B	$(1, R_{m+1}, q_n)$	q_n
			$(0, R_{m+1}, q_n)$	$1 - q_n$
	PI	B	$(1, R_m, q_{n+1})$	q_{n+1}
			$(0, R_m, q_{n+1})$	$1 - q_{n+1}$

Table II.2: Transition Function

- $q^t \in \{q_n \mid 0 < q_0 < \ldots < q_n < q_{n+1} < \ldots < q_N \leq 1\}$ refers to the entrepreneur's process reliability in period t, i.e. the probability of having to spend period t on fire-fighting is $1 - q^t$. After spending a time period on process improvement (PI), the process reliability increases from q_n to q_{n+1}. Again, a higher index denotes a better state.

The state transitions are summarized in Table II.2. In §6, we allow for more complex transitions such as random process deterioration and uncertain outcome of revenue enhancement efforts.

The entrepreneurs seek to maximize their long term expected profit. We assume risk neutrality, recognizing that other risk attitudes are natural directions for future research. We denote δ as the discrete monetary discount factor, $C(q_n)$ as a Bernoulli random variable with success probability q_n representing the availability of time (no crisis), and π as a non-anticipating time allocation policy. Thus, the objective of the entrepreneur is to maximize the following expected infinite-horizon discounted value of future net revenues:

$$E_\pi\Big[\sum_{t=0}^{\infty}\delta^{t+1}\{B+R^{t+1}(a^t)C^{t+1}(q^{t+1}(a^t))\}|R^0,q^0\Big]$$

$$=\sum_{t=0}^{\infty}B\delta^{t+1}+E_\pi\Big[\sum_{t=0}^{\infty}\delta^{t+1}R^{t+1}(a^t)C^{t+1}(q^{t+1}(a^t))|R^0,q^0\Big].$$

There exists an optimal time allocation policy π^* that is stationary (Bertsekas 2000, Proposition 7.3.1), which can be found by solving the following dynamic program,

$$\begin{aligned}
V(0,R_m,q_n) = \quad & 0+\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n), && \text{(FF)}\\
V(1,R_m,q_n) = \quad & \max\Big\{0+\delta q_{n+1}V(1,R_m,q_{n+1})+\delta(1-q_{n+1})V(0,R_m,q_{n+1}), && \text{(PI)}\\
& 0+\delta q_n V(1,R_{m+1},q_n)+\delta(1-q_n)V(0,R_{m+1},q_n), && \text{(RE)}\\
& R_m+\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n)\Big\}. && \text{(RG)}
\end{aligned} \tag{II.1}$$

4.2 Discounting Time and Return on Time Invested

We transform the discounted revenue stream with interruptions into an equivalent discounted revenue stream without interruptions by using a transformed discount factor, $\zeta(q)$, defined next.

Definition II.1 *We define the* time discount factor $\zeta(q)$ *as*

$$\zeta(q)=\frac{\delta q}{1-\delta(1-q)}.$$

The next lemma illustrates the properties of the time discount factor.

Lemma II.1 $\zeta(q)$ *is concave increasing in* $q\in[0,1]$*, with* $\zeta(0)=0$ *and* $\zeta(1)=\delta$.

The lemma shows that future time should be discounted more heavily when the frequency of interruptions $1-q$ is higher. Also note that, consistent with the behavioral studies which find that people discount future time more heavily than they discount future money (Zauberman and Lynch 2005), $\zeta(q) \leq \delta$, $\forall q$. Our first proposition shows that the 3-dimensional state DP (II.1) can be simplified to an equivalent 2-dimensional state DP using the time discount factor $\zeta(q)$.

Proposition II.1 *The DP expression (II.1) can be simplified as follows:*

$$V(R_m, q_n) = \max\left\{\zeta_{n+1}V(R_m, q_{n+1}), \quad \zeta_n V(R_{m+1}, q_n), \quad R_m + \zeta_n V(R_m, q_n)\right\}, \text{(II.2)}$$

$$\textit{where} \quad \zeta_n = \zeta(q_n).$$

From the formulation, we see that process improvement increases the time discount factor from ζ_n to ζ_{n+1} without altering the revenue rate, whereas revenue enhancement improves the revenue rate from R_m to R_{m+1} without altering the time-discount factor.

Using this observation, we can quantify the return on time invested into process improvement or revenue enhancement activities. If the entrepreneur decides to generate revenue in state (R_m, q_n) and forever after, the discounted sum of revenue is given by the expression $R_m(1+\zeta_n+\zeta_n^2+\cdots)$. If she spends her time today on process improvement to increase the process reliability from q_n to q_{n+1} and generates revenue forever after, the discounted sum of revenue is $R_m(0+\zeta_{n+1}+\zeta_{n+1}^2+\cdots)$. Finally, if she invests her time today in revenue enhancement to increase the revenue rate from R_m to R_{m+1} and generates revenue forever after, the discounted sum of revenue is $R_{m+1}(0+\zeta_n+\zeta_n^2+\cdots)$. We formalize the notion of *return on time invested* (ROTI) to gauge whether or not the time invested in process improvement or revenue enhancement offsets the opportunity cost of not generating revenue.

Definition II.2 *Suppose the entrepreneur is in state* (R_m, q_n).

(i) We define return on time invested *in* process improvement, $ROTI_n^{PI}$, *as*

$$ROTI_n^{PI} = \frac{R_m(0+\zeta_{n+1}+\zeta_{n+1}^2+\cdots)}{R_m(1+\zeta_n+\zeta_n^2+\cdots)} = \frac{\zeta_{n+1}\sum_{t=0}^{\infty}\zeta_{n+1}^t}{\sum_{t=0}^{\infty}\zeta_n^t} = \frac{\frac{\zeta_{n+1}}{1-\zeta_{n+1}}}{\frac{1}{1-\zeta_n}} = \zeta_{n+1}\frac{1-\zeta_n}{1-\zeta_{n+1}}.$$

(ii) We define return on time invested *in* revenue enhancement, $ROTI_{m,n}^{RE}$, *as*

$$ROTI_{m,n}^{RE} = \frac{R_{m+1}(0+\zeta_n+\zeta_n^2+\cdots)}{R_m(1+\zeta_n+\zeta_n^2+\cdots)} = \frac{\zeta_n\sum_{t=0}^{\infty}\zeta_n^t R_{m+1}}{\sum_{t=0}^{\infty}\zeta_n^t R_m} = \zeta_n\frac{\frac{R_{m+1}}{1-\zeta_n}}{\frac{R_m}{1-\zeta_n}} = \zeta_n\frac{R_{m+1}}{R_m}.$$

Hence, a necessary condition for the time invested in PI or RE to be earned back is that $\text{ROTI}_n^{PI} \geq 1$ or $\text{ROTI}_{m,n}^{RE} \geq 1$ respectively.

We next introduce an assumption on the sequence of revenue rates $\{R_m\}$.

Assumption II.1 *The sequence* $\{R_m\}$ *is log-concave increasing.*

In particular, Assumption II.1 holds when $\{R_m\}$ is concave increasing or when it has an s-shape, e.g. that defined by $R_{m+1} = (1+\alpha)\frac{R_m}{S}(S-R_m)\ \forall m$, where the parameters $S > R_M$ and $\alpha \in (0,1)$ are the shape parameters. In fact, growth-based entrepreneurs eventually encounter decreasing marginal return over time when increasing their revenue rate because of capacity constraints, demand constraints, or competition. For example, if the new product or service that an entrepreneur offers is not scalable, the growth in the revenue rate will eventually slow because the capacity constrains the entrepreneur. If the target market is small, the growth slows as the market becomes saturated (Cachon and Terwiesch 2009). If the entrepreneur's new product or service is highly scalable and targets a large market, her revenue rate may grow initially at an increasing rate, but will eventually slow as imitators enter the market and compete for market share (Bass 1969).

5 Analysis

In this section, we present the optimal policy for the DP (1) and elaborate on its structural results.

5.1 Optimal Policy

We first describe the necessary conditions for optimality, by characterizing the *necessary sequence* of time investments in the following lemma.

Lemma II.2 *Suppose the entrepreneur, who is in state* (R_m, q_n)*, wishes to reach state* (R_{m+i}, q_{n+j})*. Then doing j process improvements followed by i revenue enhancements dominates all other policies.*

The intuition behind the clear priority for process improvement is that it creates more time in the future by reducing the fire-fighting frequency, and that that extra time can be invested into more productive activities allowing the entrepreneur to generate revenue at a higher rate sooner.

Given this structure of the optimal policy, starting from state (R_m, q_n), identifying the optimal process reliability threshold level $q^*(R_m, q_n)$, where the entrepreneur stops process improvement and starts revenue enhancement, and the optimal revenue rate threshold level $R^*(q^*)$, where the entrepreneur stops revenue enhancement and starts generating revenue, would suffice to characterize the optimal policy. We define these threshold levels formally.

Definition II.3

(i) We define the improve-up-to level *of state* (R_m, q_n) *as the process reliability threshold level* $q^*(R_m, q_n)$*.*

(ii) We define the enhance-up-to level *of process reliability level* q^* *as the revenue rate threshold level* $R^*(q^*)$.

We now present our main result.

Theorem II.1 *(Optimal Time Allocation Policy)*
For entrepreneurs who are in state (R_m, q_n), *the optimal allocation of available time (i.e., when there is no crisis) is the following:*

If $q_n < q^*(R_m, q_n)$,

Do Process Improvement.

Else if $R_m < R^*(q^*)$,

Do Revenue Enhancement.

Else,

Do Revenue Generation.

The theorem illustrates the entrepreneur's optimal *improvement path* for maximizing the expected discounted revenue. When time is available, entrepreneurs should always first question whether or not the process reliability needs to be improved, and if so, focus on process improvement; if not, then question whether or not the revenue rate needs to be enhanced, and if so, focus on revenue enhancement; if not, only then focus on generating revenue.

The next proposition describes the properties of these threshold levels.

Proposition II.2 *Under Assumption II.1,*
(i) For any given process reliability level q_n, *the optimal improve-up-to level* $q^*(R_m, q_n)$

is nonincreasing in R_m.

(ii) For each improve-up-to level q^*, *the optimal enhance-up-to level* $R^*(q^*)$ *is nondecreasing in* q^* *and is independent of initial* R_m.

Proposition II.2 states the independence of the enhance-up-to level from the initial revenue rate R_m, but do not state the independence of the improve-up-to level from the initial process reliability q_n. We will now show that under an additional assumption on the process reliability $\{q_n\}$ and with an appropriately high discount factor δ, the independence of the improve-up-to levels can be guaranteed.

Assumption II.2 *The sequence* $\{q_n\}$ *is log-concave increasing.*

This assumption describes situations in which the entrepreneurs' efforts to increase process reliability are met with decreasing marginal returns. This can occur because improving process reliability from 50% to 55% or from 90% to 91% both require the same effort of eliminating 10% of random disturbances; and if the entrepreneurs follow the prescriptions of Pareto analysis, they should address problems from the most common to the least common. In particular, this assumption holds when the sequence $\{q_n\}$ increases exponentially, i.e. $q_n = 1 - \kappa\gamma^{\rho n}$, where $\kappa, \gamma \in (0,1)$, $\rho > 0$, and $\gamma^{\rho} < \delta$.

Furthermore, because each time period is assumed to be relatively brief (e.g. a day or a half day), the corresponding monetary discount factor δ will usually be close to 1. The next lemma illustrates the consequence of having a high discount factor.

Lemma II.3 *For any integer* $r \in \{0, \ldots, M\}$, $\exists\, \bar{\delta} < 1$ *such that* $\{(\zeta_{n+1})^r ROTI_n^{PI}\}$ *is decreasing in* n $\forall \delta \in (\bar{\delta}, 1)$.

The following proposition demonstrates the independence of the threshold $q^*(R_m, q_n)$ from the initial process reliability q_n, for each revenue rate R_m.

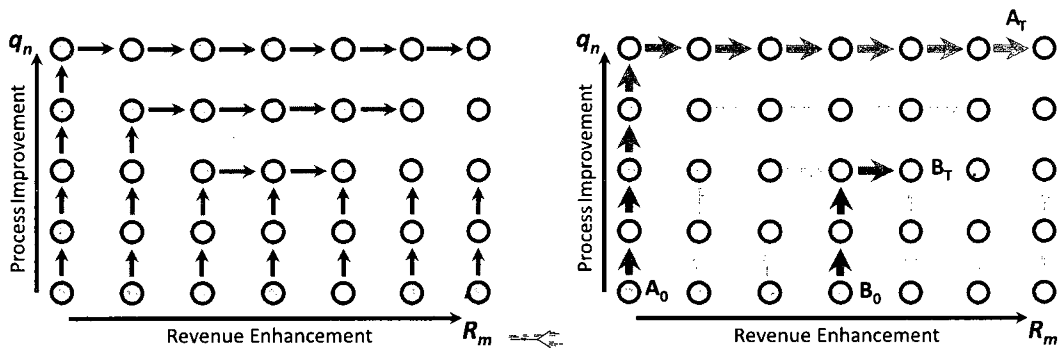

Figure II.1: Optimal Decisions. The figure on the left hand side represents the structure of the optimal policy. The figure on the right hand side compares two different optimal paths for entrepreneurs starting at different initial revenue rates.

Proposition II.3 *Suppose a given set of* $\{R_m\}$ *and* $\{q_n\}$ *satisfy Assumption II.1 and II.2 respectively, and the discount factor* δ *is such that* $\delta > \bar{\delta}$ *(Lemma II.3). Then the improve-up-to level,* $q^*(R_m)$ *is independent of the initial process reliability* q_n.

Consequently, under these assumptions, the optimal policy described in Theorem II.1 can be simplified to prescribing process improvement when $q_n < q^*(R_m)$.

A representative optimal policy is displayed in the left hand side of Figure II.1, in the two-dimensional state space constructed from a sequence of $\{R_m\}$ and $\{q_n\}$. The arrows pointing up represent process improvement decisions, the arrows pointing right represent the revenue enhancement decisions; and in states with no outgoing arrow the entrepreneur generates revenue.

5.2 Structural Properties

We discuss several properties of the optimal policy. First, all instances of the optimal policy share the common structure stated next.

Corollary II.1 *For any state (R_m, q_n), if $ROTI_n^{PI} > 1$, the optimal policy is to do process improvement.*

The corollary gives a clear prescription that, as long as the discounted sum of the future time after process improvement is greater than the discounted sum of current and future time under the status quo process, improving the process is the optimal course of action. Note that this holds regardless of the R_m, the $\text{ROTI}_{m,n}^{RE}$, or future sequences $\{R_m\}$ or $\{q_n\}$. Accordingly, one can define the minimum process reliability level $\bar{q} = \min_n \{q : \text{ROTI}_n^{PI} \leq 1\}$ below which it is optimal to do process improvement.

We now provide insights into the value of time spent on the long-term oriented activities. The fact that the optimal policy prescribes process improvement and revenue enhancement activities before revenue generation suggests that the time invested in the former activities have greater contribution to the discounted sum of revenue $V(R_m, q_n)$ than does the prevailing revenue rate R_m. For example, suppose an entrepreneur in state $(1, R_m, q_n)$ can spend her day making extra sales calls to generate another \$100 in profits. It is then common to conclude that the value of her day is \$100. However, this reasoning is wrong. Suppose she also has an opportunity to spend her day improving her processes which will save her a "net present time" of 3 days ($\text{ROTI}_n^{PI} = 3$) in the future. That means she could make at least the net present value equivalent of 3 days worth of sales calls, which is worth \$300. Thus the value of her day today should instead be \$300.

By Lemma 2 (see also Lemma II.5 in Appendix), one can express the optimal value function $V(R_m, q_n)$ in terms of the ROTI's as follows:

$$V(R_m, q_n) = \frac{R_m}{1-\zeta_n} \prod_{i=n}^{n^*-1} (\mathrm{ROTI}_i^{PI}) \prod_{j=1}^{m^*-1} (\mathrm{ROTI}_{j,n^*}^{RE}),$$

where the n^* and m^* represent the indices for the optimal improve-up-to level $q^*(R_m, q_n)$ and optimal enhance-up-to level $R^*(q^*)$ respectively. The optimal long term value of being in state (R_m, q_n) is thus the product of the discounted sum of the prevailing revenue rate in the status quo process $\frac{R_m}{1-\zeta_n}$, the return on $(n^* - n)$ time periods invested in process improvement $\prod_{i=n}^{n^*-1}(\mathrm{ROTI}_i^{PI})$, and the return on $(m^* - m)$ time periods invested in revenue enhancement $\prod_{j=m}^{m^*-1}(\mathrm{ROTI}_{i,j}^{RE})$. In other words, $V(R_m, q_n)$ equals the discounted sum of the prevailing revenue rate $\frac{R_m}{1-\zeta_n}$ only when $\mathrm{ROTI}_i^{PI} \leq 1\ \forall i$ and $\mathrm{ROTI}_{j,k}^{RE} \leq 1\ \forall j,k$, at which point the optimal policy prescribes the entrepreneur to generate revenue.

The next corollary shows that the optimal policy may prescribe entrepreneurs to improve the process reliability beyond the minimum process reliability level $\bar{q}$, even when $\mathrm{ROTI}_{m,n}^{RE} > 1$.

Corollary II.2 *It is sometimes optimal to do process improvement even when* $ROTI_n^{PI} < 1$ *and* $ROTI_{m,n}^{RE} > 1$.

The corollary states that the optimal policy prescribes process improvement even when the time invested in process improvement cannot be earned back and the revenue rate can be enhanced considerably. The intuition is as follows. When the future revenue rate is high compared to the current revenue rate R_m, the cost of investing time in process improvement in the current period can more easily be recovered by generating revenue at a higher rate $R_{m+\tau} > R_m$ after undergoing τ revenue enhancements in the future. In addition, if many time periods will be spent on revenue enhancement activities, improving beyond the minimum process reliability is optimal in order to be less often interrupted by fire-fighting during the revenue enhancement activities.

The next result shows that it is optimal for a high revenue rate-endowed entrepreneur to spend less time on process improvement and revenue enhancement relative to a lower revenue rate-endowed entrepreneur who is operating in the same market, which causes the latter to sometimes eclipse her high revenue rate counterpart and to earn revenue at a higher rate in the future. This phenomenon, described formally in Proposition II.4, is illustrated in the right hand side of Figure II.1.

Proposition II.4 *Given sequences $\{R_m\}$ and $\{q_n\}$, satisfying Assumption II.1 and II.2, and $\delta > \bar{\delta}$ (Lemma II.3), suppose there are two entrepreneurs A and B initially with revenue rates $R^0_{m_A}$ and $R^0_{m_B}$ respectively, with $R^0_{m_A} < R^0_{m_B}$ and arbitrary process reliabilities $q^0_{n_A}$ and $q^0_{n_B}$. Then, the optimal policy prescribes entrepreneurs A and B to reach revenue rates $R^T_{m_A}$ and $R^T_{m_B}$ respectively, where $R^T_{m_A} \geq R^T_{m_B}$.*

In certain situations, the sequence $\{q_n\}$ may be such that $\mathrm{ROTI}^{PI}_n > 1\ \forall n$, leading the entrepreneur to improve her process to make it as reliable as possible, i.e., $\bar{q}_n = q_N$, similar to the "quality is free" concept (Crosby 1979). In that special case, the following one-stage-look-ahead policy is optimal.

Corollary II.3 *For entrepreneurs who are in state (R_m, q_n), suppose that $\{q_n\}$ is such that $ROTI^{PI}_n > 1\ \forall n$. Then the optimal allocation of available time (when there is no need for fire-fighting) is the following one-stage-look-ahead decision rule:*

If $ROTI_n^{PI} > 1$,

Do Process Improvement.

Else if $ROTI_{m,N}^{RE} > 1$,

Do Revenue Enhancement.

Else,

Do Revenue Generation.

Corollary II.3 states that entrepreneurs can sometimes allocate their time optimally by only looking ahead one stage. In other words, the knowledge of (i) the time discount factor $\zeta(q)$ and (ii) ROTI_n^{PI} and $\text{ROTI}_{m,n}^{RE}$ may be sufficient for allocating their time optimally, and the knowledge of the future sequences $\{q_n\}$ and $\{R_m\}$ may not necessarily add value.

6 Stochastic Process Deteriorations and Revenue Enhancements

In this section, we generalize the DP model (II.1) to allow for minor perturbations in the process reliability as well as the uncertainty in the revenue enhancement efforts. First, processes frequently deteriorate if they are not consciously maintained. We incorporate this by assuming that the process reliability can deteriorate from q_n to q_{n-1} $\forall n \geq 1$ each period with probability $1-\beta$ when the entrepreneur is not engaging in a process improvement activity (the process in its worst process reliability level q_0 is assumed not to deteriorate further). We assume that this β is an exogenous variable which is independent of the state.

Second, despite knowing the sequence of future revenue rates $\{R_m\}$, realizing those higher revenue rates are often uncertain and may require multiple attempts. We model this by assuming that the revenue enhancement efforts are successful only with probability $\alpha < 1$. The sequence of stochastic events and the corresponding state transitions are described in Figure II.2.

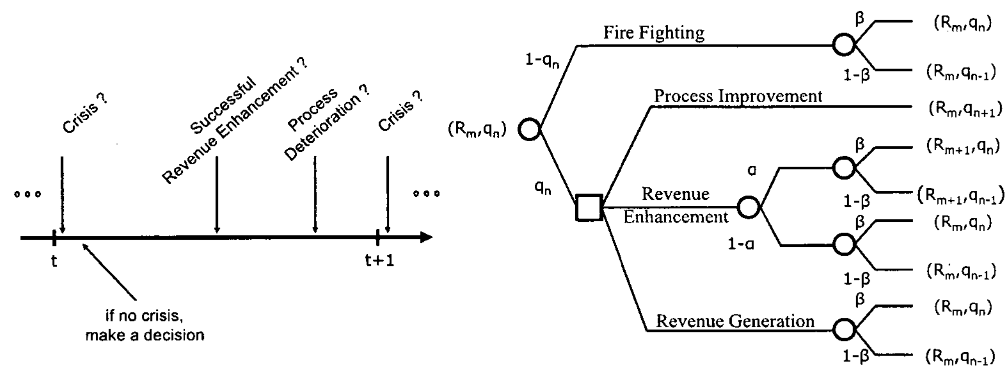

Figure II.2: Sequence of stochastic events and the corresponding state transitions

With stochastic process deteriorations and uncertain revenue enhancement efforts, the DP model (II.1) becomes:

$$
\begin{aligned}
V(0,R_m,q_n) = {} & 0+\beta\Big[\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n)\Big]+ && \text{(FF)}\\
& +(1-\beta)\Big[\delta q_{n-1}V(1,R_m,q_{n-1})+\delta(1-q_{n-1})V(0,R_m,q_{n-1})\Big] \\
V(1,R_m,q_n) = {} & \max\Big\{0+\delta q_{n+1}V(1,R_m,q_{n+1})+\delta(1-q_{n+1})V(0,R_m,q_{n+1}), && \text{(PI)}\\
& 0+\alpha\Big(\beta\Big[\delta q_n V(1,R_{m+1},q_n)+\delta(1-q_n)V(0,R_{m+1},q_n)\Big]+ && \text{(RE)}\\
& +(1-\beta)\Big[\delta q_{n-1}V(1,R_{m+1},q_{n-1})+\delta(1-q_{n-1})V(0,R_{m+1},q_{n-1})\Big]\Big), \\
& +(1-\alpha)\Big(\beta\Big[\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n)\Big]+ \\
& +(1-\beta)\Big[\delta q_{n-1}V(1,R_m,q_{n-1})+\delta(1-q_{n-1})V(0,R_m,q_{n-1})\Big]\Big), \\
& R_m+\beta\Big[\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n)\Big]+ && \text{(RG)}\\
& +(1-\beta)\Big[\delta q_{n-1}V(1,R_m,q_{n-1})+\delta(1-q_{n-1})V(0,R_m,q_{n-1})\Big]\Big\},
\end{aligned}
\tag{II.3}
$$

The next proposition states the conditions under which the expression (II.3) can be simplified to a 2-dimensional state DP using modified time discount factors.

Proposition II.5 *Suppose the entrepreneur is in state (R_m, q_n), and it is optimal to do process improvement in states $(1, R_m, q_k)$ $\forall k < n$ and in states $(1, R_{m+1}, q_k)$ $\forall k \leq n$. Then, expression (II.3) can be simplified to:*

$$V(R_m, q_n) = \{\zeta_{\beta,n+1} V(R_m, q_{n+1}),\ \zeta_n(\alpha, \beta) V(R_{m+1}, q_n),\ R_m + \zeta_n(\beta) V(R_m, q_n)\}, \quad \text{(II.4)}$$

$$\textit{where} \quad \zeta_{\beta,n} = \frac{\delta q_n}{1 - \delta(1 - q_n)(\beta + (1 - \beta)\zeta_{\beta,n-1})}, \quad \textit{with} \quad \zeta_{\beta,0} = \zeta_0, \quad \textit{and}$$

$$\zeta_n(\beta) = \beta \zeta_{\beta,n} + (1 - \beta)\zeta_{\beta,n}\zeta_{\beta,n-1}, \quad \zeta_n(\alpha, \beta) = \frac{\zeta_n(\beta)\alpha}{1 - \zeta_n(\beta)(1 - \alpha)}.$$

The new discount factors take into consideration, in addition to the chance of fire-fighting, the chance of doing process improvement when deterioration in the process occurs, as well as the failed revenue enhancement efforts. One can easily verify that all the discount factors monotonically increase and approach ζ_n as $\alpha, \beta \to 1$.

We next define the return on time invested in process improvement in the presence of stochastic process deterioration.

Definition II.4 *Suppose the entrepreneur is in state $(1, R_m, q_n)$, with deterioration probability $1 - \beta$, and it is optimal to do process improvement in states $(1, R_m, q_k)$ $\forall k < n$. We define* return on time invested *in* process improvement, *$ROTI_n^{PI}(\beta)$, as*

$$ROTI_n^{PI}(\beta) = \zeta_{\beta,n+1} \frac{1 - \zeta_n(\beta)}{1 - \zeta_{n+1}(\beta)}.$$

As in the previous section, we assume the same structure of the sequences $\{R_m\}$ and $\{q_n\}$ as stated in Assumption II.1 and II.2. Additionally, we will assume the following.

Assumption II.3

The discount factor $\delta < 1$ is large enough such that $\{ROTI_n^{PI}(\beta)\}$ decreases in n.

In the next section, we will restrict our attention to the sequences of $\{R_m\}$, $\{q_n\}$, and the discount factor δ which satisfy Assumptions II.1–II.3, and characterize the structural properties of the optimal policy.

6.1 Optimal Policy

We have the following necessary conditions for optimality, which is an extension of Lemma II.2 to accommodate stochastic transitions.

Lemma II.4 *Suppose the optimal decisions in states (R_{m+1}, q_k) $\forall k \leq n$ and (R_m, q_k) $\forall k < n$ are process improvement. Then, an entrepreneur who went from state (R_m, q_n) to state (R_{m+1}, q_{n+1}) did process improvement in state (R_m, q_n).*

The next proposition describes the properties of the resulting optimal policy.

Proposition II.6 *Let $\bar{q} = \min_n \{q : ROTI_n^{PI}(\beta) \leq 1\}$.*
(i) For any process reliability level $q_n \leq \bar{q}$, the optimal improve-up-to level $q^*(R_m, q_n)$ is nonincreasing in R_m,
(ii) For each improve-up-to level q^*, the optimal enhance-up-to level $R^*(q^*)$ is nondecreasing in q^* and is independent of initial R_m.

For any state $(1, R_m, q_n)$, $q_n \leq \bar{q}$, as the properties of the improve-up-to level and the enhance-up-to level stipulated in Proposition II.6 remain identical to that stipulated in Proposition II.2, all the corollaries from the previous section, as well as Proposition II.4, carry over to the stochastic case. Moreover, the structure of Theorem II.1 generalizes to the case in which the process is allowed to stochastically deteriorate. In particular, if the process stochastically deteriorates while engaging in revenue enhancement or revenue generation activities, the theorem prescribes that the entrepreneur should restore the process to the necessary improve-up-to level before engaging further in those activities. However, if the process reliability level remains above the improve-up-to level after deterioration, the entrepreneur should not improve the process, causing the enhance-up-to levels to be altered. Finally, if the revenue enhancement effort is unsuccessful, the optimal policy prescribes the entrepreneur to try again until she reaches the appropriate enhance-up-to level.

In the next section, we numerically illustrate the optimal policy of the general DP model (II.3) starting from state with process reliability level $q_n \leq \bar{q}$.

6.2 Illustration of the Optimal Policy

Suppose that the revenue sequence is increasing in an s-shaped fashion, specifically $R_{m+1} = 1.2\frac{R_m}{100}(100 - R_m)$ and that the process reliability sequence is concave increasing, specifically, $q_n = 1 - (0.75)^n$. Further suppose that the uncertain revenue enhancement parameter $\alpha = 0.5$ and that the stochastic process deterioration parameter $\beta = 0.9$. Moreover, we assume a reasonably high discount factor $\delta = 0.95$. For these particular sequences, one can easily verify that Assumptions II.1–II.3 hold.

A representative sample path of the optimal policy starting from $(R^0, q^0) = (2, 0.25)$, is shown in Figure II.3. Similar to Figure II.1, Figure II.3 depicts the optimal improvement path in the two dimensional state space. We distinguish three phases: the process improvement phase, the revenue enhancement phase, and the revenue generation phase. The process improvement phase is characterized by process improvements until the process reaches the improve-up-to level associated with the initial revenue rate

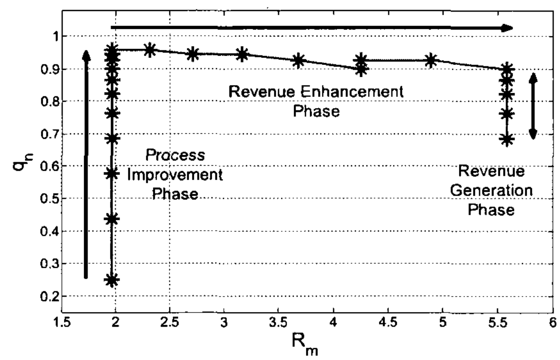

Figure II.3: An illustration of the sample path in the state space, and its temporal dynamics. The sample paths consists of three phases: PI, RE, and RG phases.

$q^*(R^0, q^0)$. Note that the process reliability level may not monotonically increase in the PI phase due to stochastic deteriorations. During the revenue enhancement phase, the entrepreneur invests time in revenue enhancement (with uncertain outcome), and the process may deteriorate at any time. During this phase, the optimal policy prescribes that the entrepreneur should improve the process as soon as it deteriorates to a process reliability below the prevailing improve-up-to level. Observe that with stochastic process deteriorations, the prevailing improve-up-to level may actually be lower than the initial improve-up-to level $q^*(R^0, q^0)$ by Proposition II.6(i). After reaching the enhance-up-to level $\bar{R} \equiv R^*(q)$, the entrepreneur starts the revenue generation phase. Observe that, by Proposition II.6(ii), the final revenue rate $\bar{R}$ can be lower than the initially targeted revenue rate $R^*(q^*(R^0))$ due to stochastic process deterioration. In that last phase, the improve-up-to level $q^*(\bar{R})$ is equal to the minimum process reliability level $\bar{q}$. In other words, without any revenue enhancement opportunities, it does not pay to engage in process improvement when the time invested in process improvement cannot be earned back. Thus, the entrepreneur generates revenue at a constant rate,

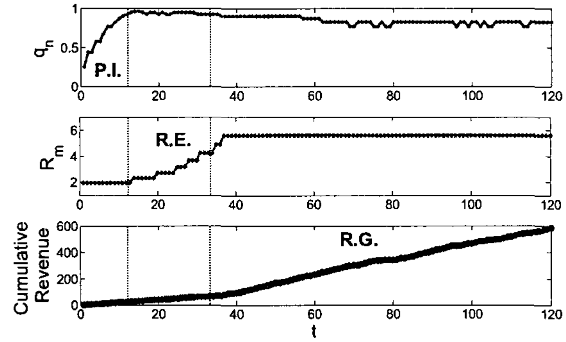

Figure II.4: Temporal evolution of process reliability q_n, revenue rate R_m, and the cumulative revenue.

$\bar{R}$ (provided there is no crisis) and continues to focus on revenue generation until the process deteriorates below $\bar{q}$, only after which she improves the process back to $\bar{q}$.

The evolution of the process reliability, revenue rate, and the cumulative revenue is shown in Figure II.4. During the process improvement phase, the process reliability is increased to the appropriate improve-up-to level, which is above the minimum process reliability level $\bar{q}$. In other words, it is optimal to *over-improve* the process upfront rather than later to create a *safety stock* of time.

In both the revenue enhancement and revenue generation phases, the process is improved as soon as it deteriorates to a level below the prevailing improve-up-to level. Furthermore, during the revenue enhancement phase, we can observe the uncertainty in the revenue enhancement efforts as the revenue is not always enhanced immediately, but only after multiple attempts. Once the entrepreneur reaches the enhance-up-to level and enters the revenue generation phase, she starts generating revenue at the higher revenue rate, which is represented by the increased slope in the cumulative revenue.

Finally, a sample path and the temporal dynamics illustrating Proposition II.4 are shown in Figure II.5. Consistent with Proposition II.4, Figure II.5 illustrates that en-

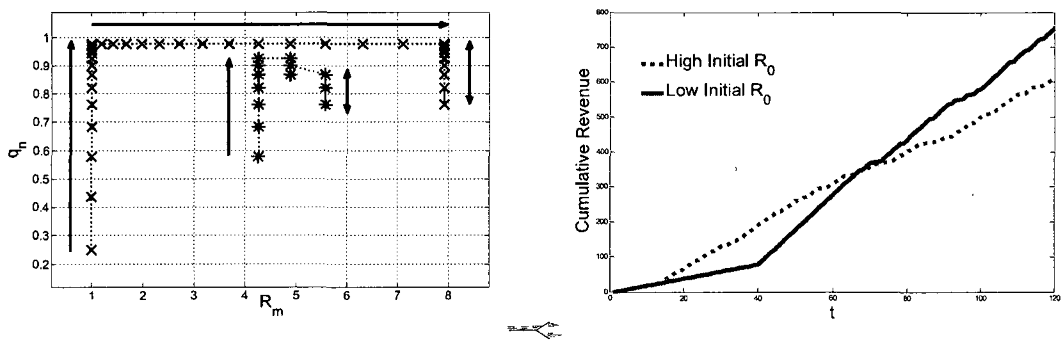

Figure II.5: Different sample paths. One with a low revenue rate, one with a high revenue rate.

trepreneurs who start from a low revenue rate should invest more time in both process improvement and revenue enhancement than their counterparts who start from a high revenue rate. This causes the former to start accumulating revenue later but at a faster rate than the latter.

7 Practical Time Management Behaviors

In this section, within the context of our model, we illustrate why common time management traps are suboptimal. In particular, we highlight the performance difference between the optimal policy and two common time management behaviors: "fix-it-later" and "growth-before-process."

Fix-it-later. Under the "fix-it-later" policy, an entrepreneur who is in her revenue generation phase with an improve-up-to level $\bar{q}$, will not improve the process immediately after it deteriorates as prescribed by the optimal policy, but instead decides to "fix it later" until the process deteriorates to an intolerable level. From that point, she dedicates her time to process improvement until she reaches the improve-up-to level, $\bar{q}$. Entrepreneurs may employ this heuristic due to procrastination (O'Donoghue and Rabin 1999), or because they feel that they do not have enough time to divert from revenue generation to process improvement (Repenning and Sterman 2002).

We use the parameters $\beta = 0.85$, $\delta = 0.98$, and the sequence $\{q_n\}$ defined by $q_n = 1-(0.75)^n$, where $\bar{q} = 0.94$, and $R_m = 10\ \forall m$ to illustrate the dynamics. The sample paths comparing the optimal policy and the "fix-it-later" policy are shown in Figure II.6. The process reliability levels of the two policies are characterized on the

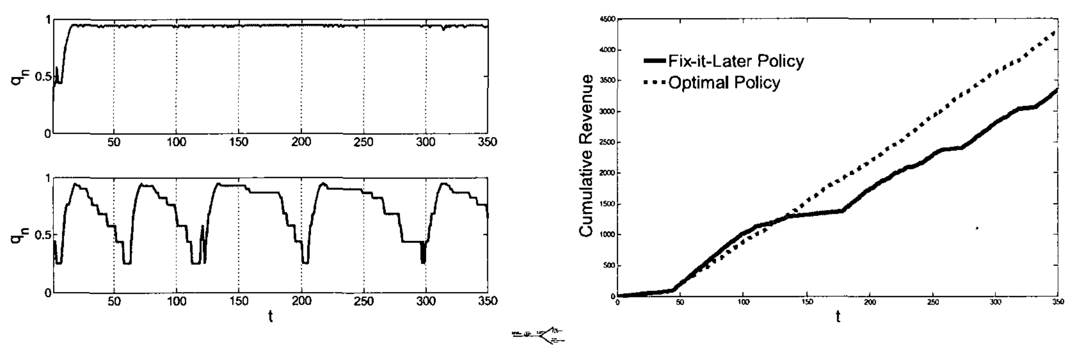

Figure II.6: Difference in sample paths: immediate process restoration (optimal policy) vs. fix it later heuristic.

left hand side, with the heuristic (bottom figure) showing much larger dips in process reliability than the optimal policy (top figure). In fact, under the fix-it-later policy, the process reliability evolves in a sawtooth pattern over time. From the cumulative revenue on the right hand side, we observe that the heuristic is initially slightly more attractive as it generates more revenue by not diverting time to process improvement.

However, delaying the process improvement causes the process to deteriorate further. Consequently, the entrepreneur must devote significantly more time fire-fighting, which causes the revenue accumulation to slow, as reflected by the decreasing slope in the cumulative revenue. Thus, even though the practice of immediate fix (process improvement) at the expense of revenue generation may seem costly initially, by doing so, entrepreneurs can earn substantially more in revenue over the long run, resulting in, for this example, an additional 12% gain in discounted revenue.

Growth-before-process. Under the "growth-before-process" policy, an entrepreneur who is in her revenue enhancement phase prefers to focus solely on revenue enhancement and to neglect process improvement until the revenue rate is high enough or the process reliability deteriorates to an intolerable level. This is common to entrepreneurs as they want to expand and reach a higher revenue rate as soon as possible, or because revenue enhancement often gives higher return than process improvement (e.g. $\mathrm{ROTI}^{RE}_{m,n} > \mathrm{ROTI}^{PI}_{n}$).

We use the parameters $\alpha = 0.75$, $\beta = 0.9$, $\delta = 0.95$, and the truncated sequence $\{q_n\}$ defined by $q_n = 1 - (0.75)^n$, $n \in \{1,2,\ldots,6\}$, and $\{R_m\}$ defined by $R_{m+1} = 1.2\frac{R_m}{100}(100 - R_m)$. Consequently, we have that $\mathrm{ROTI}^{PI}_5(\beta) > 1$, and hence the optimal improve-up-to level for all R_m is $q_6 = 0.82$, as in Corollary II.3. We let $(R^0, q^0) = (1, 0.82)$, so that the entrepreneur has finished the PI phase and is entering the RE phase. Figure II.7 compares the sample paths of this heuristic with that of the optimal policy.

We notice from the figure that prioritizing on revenue enhancement rather than process improvement in the revenue enhancement phase can cause the entrepreneurs to reach the final revenue rate *later* than they otherwise would by prioritizing on process improvement. This is because focusing on revenue enhancement and allowing the process to deteriorate causes increasingly more eruption of crises, limiting the entrepreneur's time to invest in further revenue enhancement activities. Focusing on the process improvement, on the other hand, limits such time wasted on fire-fighting, and allows the entrepreneur to spend more time on revenue enhancement (as opposed to fire-fighting) and to realize her desired revenue rate sooner, resulting in, for this example, an additional 16% gain in discounted revenue.

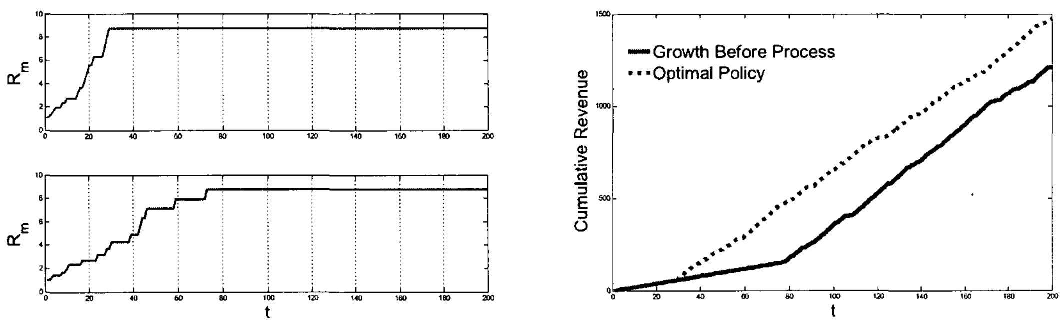

Figure II.7: Differences in the sample paths: process-focused (optimal policy) vs. growth-before-process heuristic

8 Concluding Remarks

Time is often the most constrained resource for an entrepreneur. In this paper, we develop a stylized time allocation model and provide clear guiding principles for time management to help growth-focused entrepreneurs avoid costly time management blunders. We hope that this work may serve as a building block for future research in entrepreneurial operations. We now summarize our four major findings.

First, the optimal policy prescribes entrepreneurs to make process improvement their top priority, and in particular, to maintain the process reliability at the appropriate level before engaging in other activities. According to our model, investing in process improvement creates more time in the future by reducing the time spent on random interruptions, allowing the entrepreneurs to reach their growth target faster and to generate more revenue than any other policy.

Second, entrepreneurs should seek to overinvest in process improvement relative to their long-term reliability target, i.e. create a safety stock of time upfront, when they foresee many revenue enhancement opportunities. This is because when the future target revenue rate is high compared to the current revenue rate, the cost of investing time in process improvement in the current period can more easily be recovered by generating revenue at a higher rate in the future. Moreover, by doing so, they can be distracted less often by fire-fighting during the long phase of revenue enhancement activities.

Third, our model introduces a framework for evaluating the opportunity cost of time. In particular, the opportunity cost of time should not be equated with the prevailing revenue rate, as is commonly done in practice. Instead, one should consider the impact her time invested in the process improvement or revenue enhancement activities has on her future stream of available time, i.e. the ROTI's, as such use of time may bring greater long term value than is indicated by the prevailing revenue rate. The opportunity cost of time is equal to the prevailing revenue rate only when there is no opportunity for long term improvement.

Finally, the optimal policy prescribes that an entrepreneur with a high initial revenue rate should invest less time in process improvement and in revenue enhancement activities compared to an entrepreneur with a low initial revenue rate. In other words, for the market leader who is able to generate revenue at a high rate, it is optimal to be myopic and to harvest the revenue earlier (as improving the process or enhancing the revenue stream comes at the expense of reaping large short-term rewards), whereas for the market follower, it is optimal to invest time in more long term oriented activities. This prescription sometimes results in the follower overtaking the market leader and accumulating revenue at a higher rate in the future.

There are many issues that we have not explored here. For instance, the model ignores a dynamic cash constraint. In the presence of a cash constraint, the entrepreneur would need to generate revenue, even though the process improvement target or the revenue enhancement target have not been reached. It would be interesting to see if the dominance of the PI activities over RE activities would remain valid, but we leave this topic for future research. We could also expand the set of activities of the entrepreneurs to include relations with donors, which could increase the entrepreneur's capital for a certain time horizon. We could also relax the assumption that the existence of revenue generation and revenue enhancement opportunities are constant. Furthermore, the model currently assumes that the time intervals between decision making epochs are identical and that there is no loss of time associated with switching between activities, hence ignoring the potential benefits of consolidating time into larger blocks. The duration of a crisis could also be considered stochastic, although we conjecture that the insights would remain the same. Generalization of some of our results to address such issues, although challenging, would be worthwhile.

Appendix

Proof of Lemma II.1. First, from the expression, it is clear that $\zeta(0) = 0$ and $\zeta(1) = \delta$. By taking the first derivative, we have

$$\frac{\partial\zeta(q)}{\partial q} = \frac{(1-\delta(1-q))\delta - \delta^2 q}{(1-\delta(1-q))^2} = \frac{\delta(1-\delta)}{(1-\delta(1-q))^2} > 0,\ \forall\, \delta \in (0,1).$$

Taking the second derivative, we have,

$$\frac{\partial}{\partial q}\Big(\frac{\delta(1-\delta)}{(1-\delta(1-q))^2}\Big) = -\frac{2\delta^2(1-\delta)(1-\delta(1-q))}{(1-\delta(1-q))^4} < 0,\ \forall\, \delta, q \in (0,1).$$

■

Proof of Proposition II.1. Substituting the expression of $V(0,R_m,q_n)$ from expression (FF) into the expression for $V(1,R_m,q_n)$ in equation (II.1), we get

$$V(1,R_m,q_n) = \max\Big\{0+V(0,R_m,q_{n+1}),\ 0+V(0,R_{m+1},q_n),\ R_m+V(0,R_m,q_n)\Big\} \tag{II.5}$$

Furthermore, from (FF) in (II.1), we observe the following relationship between $V(0,R_m,q_n)$ and $V(1,R_m,q_n)$:

$$V(0,R_m,q_n) = \frac{\delta q_n}{1-\delta(1-q_n)} V(1,R_m,q_n) = \zeta_n V(1,R_m,q_n).$$

Substituting this expression into equation (II.5), we can rewrite them solely in terms of the non-crisis states, and can drop the redundant "1" from the state variable. ■

Proof of Lemma II.2. We apply the interchange argument (Bertsekas 2000, §4.5) to show that the optimal policy prescribes j consecutive process improvement followed by i consecutive revenue enhancements. Suppose the entrepreneur engages in i RE's and j PI's in the arbitrary order (RE, PI, $\cdots$, RE, PI, $\cdots$, PI, RE) to reach the target state (R_{m+i}, q_{n+j}), where she starts generating revenue. The expected net present value of the resulting revenue generated is:

$$\frac{R_m}{1-\zeta_n} \cdot \text{ROTI}^{RE}_{m,n} \cdot \text{ROTI}^{PI}_{n} \cdots \text{ROTI}^{PI}_{n+j-1} \cdot \text{ROTI}^{RE}_{m+i-1,n+j}.$$

By changing the order of any two consecutive activities of (*RE*, *PI*) to (*PI*, *RE*) at any point in the sequence, we increase the expected value from $\text{ROTI}^{RE}_{m,n} \cdot \text{ROTI}^{PI}_{n} = \zeta_n \frac{R_{m+1}}{R_m} \cdot \zeta_{n+1} \frac{1-\zeta_n}{1-\zeta_{n+1}}$ to $\text{ROTI}^{PI}_{n} \cdot \text{ROTI}^{RE}_{m,n+1} = \zeta_{n+1} \frac{1-\zeta_n}{1-\zeta_{n+1}} \cdot \zeta_{n+1} \frac{R_{m+1}}{R_m}$, where $\zeta_{n+1} > \zeta_n$ by Lemma II.1. ■

Proof of Theorem II.1. Because the horizon is infinite and the rewards are bounded, there exists an optimal policy that is stationary (Bertsekas 2000, Proposition 7.3.1). Furthermore, by Lemma II.2, the optimal policy is to do process improvement as long as $q_n < q^*(R_m, q_n)$, then do revenue enhancement as long as $R_m < R^*_m(q_n)$, and only then do revenue generation.

■

Lemma II.5 *Under Assumption II.1,*

$$V(R_m, q_n) = \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \zeta_{n+j^*(m,n)}^{i^*(m,n+j^*(m,n))} \frac{R_{m+i^*(m,n+j^*(m,n))}}{1-\zeta_{n+j^*(m,n)}}, \tag{II.6}$$

where

$$i^*(m,n) = \max\left\{ i > 0 : \zeta_n \frac{R_{m+i}}{R_{m+i-1}} > 1 \right\}, \tag{II.7}$$

$$j^*(m,n) = \max\left\{ j > 0 : \prod_{k=1}^{j} \left(\zeta_{n+k} \frac{1-\zeta_{n+k-1}}{1-\zeta_{n+k}} \frac{\zeta_{n+k}^{i^*(m,n+k)} R_{m+i^*(m,n+k)}}{\zeta_{n+k-1}^{i^*(m,n+k-1)} R_{m+i^*(m,n+k-1)}} \right) \right\} \tag{II.8}$$

Proof of Lemma II.5. Applying the interchange argument in Lemma II.2 to equation (II.2), we have

$$V(R_m, q_n) = \max_{j=0,\ldots,N-n} \prod_{j'=1}^{j} \zeta_{n+j'} \max_{i=0,\ldots,M-m} \left\{ \zeta_{n+j}^{i} \frac{R_{m+i}}{1-\zeta_{n+j}} \right\}. \tag{II.9}$$

For any j, note that

$$\max_{i=0,\ldots,M-m}\left\{i:\zeta_{n+j}^{i}\frac{R_{m+i}}{1-\zeta_{n+j}}\right\} = \max_{i=0,\ldots,M-m}\left\{i:\prod_{i'=0}^{i}\zeta_{n+j}\frac{R_{m+i'}}{R_{m+i'-1}}\right\}\frac{R_{m-1}}{1-\zeta_{n+j}}$$
$$= \max_{i=0,\ldots,M-m}\left\{i>0:\zeta_{n+j}\frac{R_{m+i}}{R_{m+i-1}}>1\right\},$$

where the first equality is by the telescoping product, and the second equality is by Assumption II.1. Thus we have the expression for $i^*(m,n+j)$. Using the $i^*(m,n+j)$, equation (II.9) consequently becomes,

$$V(R_m,q_n) = \max_{j=0,\ldots,N-n}\prod_{j'=1}^{j}\zeta_{n+j'}\zeta_{n+j}^{i^*(m,n+j)}\frac{R_{m+i^*(m,n+j)}}{1-\zeta_{n+j}}. \tag{II.10}$$

Note that

$$\max_{j=0,\ldots,N-n}\left\{j:\prod_{j'=1}^{j}\zeta_{n+j'}\zeta_{n+j}^{i^*(m,n+j)}\frac{R_{m+i^*(m,n+j)}}{1-\zeta_{n+j}}\right\}$$
$$= \max_{j=0,\ldots,N-n}\left\{j:\prod_{k=1}^{j}\left(\frac{\prod_{j'=1}^{k}\zeta_{n+j'}\zeta_{n+k}^{i^*(m,n+k)}\frac{R_{m+i^*(m,n+k)}}{1-\zeta_{n+k}}}{\prod_{j'=1}^{k-1}\zeta_{n+j'}\zeta_{n+k-1}^{i^*(m,n+k-1)}\frac{R_{m+i^*(m,n+k-1)}}{1-\zeta_{n+k-1}}}\right)\right\}\left(\zeta_{n}^{i^*(m,n)}\frac{R_{m+i^*m,n}}{1-\zeta_{n}}\right)$$
$$= \max_{j=0,\ldots,N-n}\left\{j>0:\prod_{k=1}^{j}\left(\zeta_{n+k}\frac{1-\zeta_{n+k-1}}{1-\zeta_{n+k}}\frac{\zeta_{n+k}^{i^*(m,n+k)}R_{m+i^*(m,n+k)}}{\zeta_{n+k-1}^{i^*(m,n+k-1)}R_{m+i^*(m,n+k-1)}}\right)\right\}, \tag{II.11}$$

where the first equality is the telescoping product, and the second equality is after simplification. Thus we have the expression for $j^*(m,n)$. ■

Proof of Proposition II.2. The proof uses Lemma II.5, which appears in the Appendix.

(i) To show that the improve-up-to level $q^*(R_m,q_n)$ is nonincreasing in R_m, we will show that if it is optimal to do process improvement in state (R_m,q_n), it is optimal to do process improvement in all states (R_k,q_n), $\forall k<m$. In other words, for all n, using (II.2),

$$V(R_m,q_n)=\zeta_{n+1}V(R_m,q_{n+1}) \quad\Rightarrow\quad V(R_{m-1},q_n)=\zeta_{n+1}V(R_{m-1},q_{n+1}).$$

We prove by contradiction.

(a) Suppose revenue enhancement is the optimal decision in state (R_{m-1}, q_n), i.e., $V(R_{m-1}, q_n) = \zeta_n V(R_m, q_n) > \zeta_{n+1} V(R_{m-1}, q_{n+1})$. Then, using (II.6), this is equivalent to,

$$\zeta_n \zeta_{n+1} V(R_m, q_{n+1}) = \zeta_n V(R_m, q_n) > \zeta_{n+1} V(R_{m-1}, q_{n+1}) \geq \zeta_{n+1} \zeta_{n+1} V(R_m, q_{n+1})$$

where the last inequality is by equation (II.2). We have a contradiction since $\zeta_n \leq \zeta_{n+1}$ by Lemma II.1.

(b) Suppose revenue generation is the optimal decision in state (R_{m-1}, q_n), i.e., $V(R_{m-1}, q_n) = \frac{R_{m-1}}{1-\zeta_n} > \zeta_{n+1} V(R_{m-1}, q_{n+1})$. Then, by equation (II.6), we have

$$\begin{aligned} V(R_{m-1}, q_n) &= \frac{R_{m-1}}{1-\zeta_n} > \zeta_{n+1} V(R_{m-1}, q_{n+1}) \\ &= \prod_{j'=1}^{j^*(m-1,n)} \zeta_{n+j'} \left\{ \zeta_{n+j^*(m-1,n)}^{i^*(m-1,n+j^*(m-1,n))} \frac{R_{(m-1)+i^*(m-1,n+j^*(m-1,n))}}{1-\zeta_{n+j^*(m-1,n)}} \right\} \\ &\geq \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \zeta_{n+j^*(m,n)} \left\{ \zeta_{n+j^*(m,n)}^{i^*(m,n+j^*(m,n))} \frac{R_{m+i^*(m,n+j^*(m,n))}}{1-\zeta_{n+j^*(m,n)}} \right\}. \end{aligned}$$

Dividing both sides by $\frac{R_{m-1}}{1-\zeta_n}$, we have

$$\begin{aligned} 1 &> \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \frac{1-\zeta_n}{1-\zeta_{n+j^*(m,n)}} \left(\zeta_{n+j^*(m,n)} \frac{R_m}{R_{m-1}} \right) \\ &\quad \times \left\{ \zeta_{n+j^*(m,n)}^{i^*(m,n+j^*(m,n))} \frac{R_{m+i^*(m,n+j^*(m,n))}}{R_m} \right\} \\ &= \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \frac{1-\zeta_n}{1-\zeta_{n+j^*(m,n)}} \left(\zeta_{n+j^*(m,n)} \frac{R_m}{R_{m-1}} \right) \\ &\quad \times \prod_{k=1}^{i^*(m,n+j^*(m,n))} \left(\zeta_{n+j^*(m,n)} \frac{R_{m+k}}{R_{m-1+k}} \right) \\ &> \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \frac{1-\zeta_n}{1-\zeta_{n+j^*(m,n)}} \prod_{k=1}^{i^*(m,n+j^*(m,n))} \left(\zeta_{n+j^*(m,n)} \frac{R_{m+k}}{R_{m-1+k}} \right) \\ &= \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \frac{1-\zeta_n}{1-\zeta_{n+j^*(m,n)}} \left\{ \zeta_{n+j^*(m,n)}^{i^*(m,n+j^*(m,n))} \frac{R_{m+i^*(m,n+j^*(m,n))}}{R_m} \right\}, \end{aligned}$$

where the second inequality is due to Assumption II.1 and equation (II.7), i.e.,

$$\zeta_{n+j^*(m,n)} \frac{R_m}{R_{m-1}} \geq \zeta_{n+j^*(m,n)} \frac{R_{m+1}}{R_m} \geq \cdots \geq \zeta_{n+j^*(m,n)} \frac{R_{m+i^*(m,n)}}{R_{m+i^*(m,n)-1}} > 1.$$

Multiplying both sides by $\frac{R_m}{1-\zeta_n}$, and using (II.6) we have

$$\frac{R_m}{1-\zeta_n} > \prod_{j'=1}^{j^*(m,n)} \zeta_{n+j'} \left\{ \zeta_{n+j^*(m,n)}^{i^*(m,n+j^*(m,n))} \frac{R_{m+i^*(m,n+j^*(m,n))}}{1-\zeta_{n+j^*(m,n)}} \right\} = V(R_m, q_n) \geq \frac{R_m}{1-\zeta_n},$$

where the last inequality is by (II.2). This is a contradiction.

(ii) To show that the enhance-up-to level $R_m^*(q_n)$ is nondecreasing in q_n, we show that $i^*(m,n)$ defined by (II.7) is nondecreasing in n. For any $k > 0$, we have

$$\begin{aligned} i^*(m,n+k) &= \max\left\{ i \geq 0 : \zeta_{n+j^*(m,n+k)} \frac{R_{m+i}}{R_{m+i-1}} > 1 \right\} \\ &\geq \max\left\{ i \geq 0 : \zeta_{n+j^*(m,n)} \frac{R_{m+i}}{R_{m+i-1}} > 1 \right\} = i^*(m,n), \end{aligned} \quad \text{(II.12)}$$

since $\zeta(q)$ is increasing by Lemma II.1. Moreover, by Assumption II.1, for any given process reliability level q_n, the largest revenue rate $R_m^*(R_m, q_n) = R_{m^*}$ for which $\zeta_n \frac{R_{m^*}}{R_{m^*-1}} > 1$ is identical independent of the starting revenue rate. ■

Proof of Lemma II.3 We now want to show that $\exists\bar{\delta}$ such that $\forall\delta \in (\bar{\delta}, 1)$,

$$\begin{aligned}
\zeta_{k+1}^{r}\mathrm{ROTI}_{k}^{PI} &\geq \zeta_{k+2}^{r}\mathrm{ROTI}_{k+1}^{PI} \qquad \text{(II.13)} \\
&\Leftrightarrow \frac{\zeta_{k+1}\frac{1-\zeta_{k}}{1-\zeta_{k+1}}}{\zeta_{k+2}\frac{1-\zeta_{k+1}}{1-\zeta_{k+2}}} \geq \left(\frac{\zeta_{k+2}}{\zeta_{k+1}}\right)^{r} \\
&\Leftrightarrow \frac{q_{k+1}(1-\delta(1-q_{k+1}))}{q_{k+2}(1-\delta(1-q_{k}))} \geq \left(\frac{q_{k+2}(1-\delta(1-q_{k+1}))}{q_{k+1}(1-\delta(1-q_{k+2}))}\right)^{r}. \qquad \text{(II.14)}
\end{aligned}$$

For notational simplicity, we define

$$LHS \equiv \frac{q_{k+1}(1-\delta(1-q_{k+1}))}{q_{k+2}(1-\delta(1-q_{k}))} \quad \text{and} \quad RHS \equiv \left(\frac{q_{k+2}(1-\delta(1-q_{k+1}))}{q_{k+1}(1-\delta(1-q_{k+2}))}\right)^{r}.$$

First, if $\delta = 0$ and $\delta = 1$, the inequality (II.14) respectively reduces to

$$\frac{q_{k+1}}{q_{k+2}} \leq \left(\frac{q_{k+2}}{q_{k+1}}\right)^{r}, \quad \text{and} \quad \frac{q_{k+1}^{2}}{q_{k+2}q_{k}} \geq 1^{r} \Leftrightarrow \frac{q_{k+1}}{q_{k}} \geq \frac{q_{k+2}}{q_{k+1}}.$$

The first inequality holds because q_k is increasing and r is any natural number. The second inequality is true because $\{q_k\}$ is log-concave increasing (Assumption II.2).

Thus, we have our result if we show that as δ increases from 0 to 1, the LHS increases and the RHS decreases by taking the first derivatives.

$$\begin{aligned}
\frac{\partial}{\partial\delta}LHS &= \frac{-(q_{k+2}-q_{k+2}(1-q_{k})\delta)q_{k+1}(1-q_{k+1})}{(q_{k+2}-q_{k+2}(1-q_{k})\delta)^{2}} \\
&\quad + \frac{(q_{k+1}-q_{k+1}(1-q_{k+1})\delta)q_{k+2}(1-q_{k})}{(q_{k+2}-q_{k+2}(1-q_{k})\delta)^{2}} \\
&= \frac{q_{k+1}q_{k+2}\{(1-q_{k})-(1-q_{k+1})\}}{(q_{k+2}-q_{k+2}(1-q_{k})\delta)^{2}} \\
&= \frac{q_{k+1}q_{k+2}(q_{k+1}-q_{k})}{(q_{k+2}-q_{k+2}(1-q_{k})\delta)^{2}} \geq 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial\delta}RHS &= r(\text{RHS})^{r-1}\cdot\frac{-(q_{k+1}-q_{k+1}(1-q_{k+2})\delta)q_{k+2}(1-q_{k+1})}{(q_{k+1}-q_{k+1}(1-q_{k+2})\delta)^2} \\
&\quad +r(\text{RHS})^{r-1}\cdot\frac{(q_{k+2}-q_{k+2}(1-q_{k+1})\delta)q_{k+1}(1-q_{k+2})}{(q_{k+1}-q_{k+1}(1-q_{k+2})\delta)^2} \\
&= r(\text{RHS})^{r-1}\cdot\frac{q_{k+1}q_{k+2}\{(1-q_{k+2})-(1-q_{k+1})\}}{(q_{k+2}-q_{k+2}(1-q_k)\delta)^2} \\
&= r(\text{RHS})^{r-1}\cdot\frac{q_{k+1}q_{k+2}(q_{k+1}-q_{k+2})}{(q_{k+2}-q_{k+2}(1-q_k)\delta)^2}\leq 0.
\end{aligned}$$

Thus, $\exists\,\bar{\delta}$ such that $LHS \geq RHS$, i.e., the inequality (II.14) holds, if and only if $\delta \geq \bar{\delta}$.

■

Lemma II.6 *The sequence* $\left\{\zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}}\frac{\zeta_{k+1}^{i^*(m,k+1)}R_{m+i^*(m,k+1)}}{\zeta_k^{i^*(m,k)}R_{m+i^*(m,k)}}\right\}$ *is decreasing in k when* $\delta > \bar{\delta}$ *(see Lemma II.3).*

Proof of Lemma II.6. By Lemma II.3, when $\delta > \bar{\delta}$,

$$\zeta_{n+1}^r\text{ROTI}_n^{PI} \geq \zeta_{n+2}^r\text{ROTI}_{n+1}^{PI} \Rightarrow \zeta_{n+1}^{i^*(m,n+2)-i^*(m,n)}\text{ROTI}_n^{PI} \geq \zeta_{n+2}^{i^*(m,n+2)-i^*(m,n)}\text{ROTI}_{n+1}^{PI}$$

$$\begin{aligned}
&\Rightarrow\quad \zeta_{n+1}^{i^*(m,n+2)-i^*(m,n)}\text{ROTI}_n^{PI}\left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \geq \zeta_{n+2}^{i^*(m,n+2)-i^*(m,n)}\text{ROTI}_{n+1}^{PI}\left(\frac{\zeta_{n+2}}{\zeta_{n+1}}\right)^{i^*(m,n)} \\
&\Leftrightarrow\quad \text{ROTI}_n^{PI}\left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \geq \text{ROTI}_{n+1}^{PI}\left(\frac{\zeta_{n+2}}{\zeta_{n+1}}\right)^{i^*(m,n+2)},
\end{aligned}$$

where the first implication is because $i^*(m,n+1) \geq i^*(m,n)\ \forall n$ by equation (II.12), and the second implication is because $\left\{\frac{\zeta_{n+1}}{\zeta_n}\right\}$ is decreasing (Lemma II.1), and the final equivalence is after dividing both sides by $\zeta_{n+1}^{i^*(m,n+2)-i^*(m,n)}$.

Expanding the expressions using the telescoping product, we have

$$\frac{\zeta_{n+1}^{i^*(m,n+1)}R_{m+i^*(m,n+1)}}{\zeta_n^{i^*(m,n)}R_{m+i^*(m,n)}}$$

$$
\begin{aligned}
&= \frac{\zeta_{n+1}^{\frac{R_{m+1}}{R_m}} \cdot \zeta_{n+1}^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_{n+1}^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}} \cdots \zeta_{n+1}^{\frac{R_{m+i^*(m,n+2)}}{R_{m+i^*(m,n+2)-1}}}}{\zeta_n^{\frac{R_{m+1}}{R_m}} \cdot \zeta_n^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_n^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}}} \\
&= \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \zeta_{n+1}^{i^*(m,n+1)-i^*(m,n)} \frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n)}} \\
&\geq \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)},
\end{aligned}
$$

where the second term is due to equation (II.7), and

$$
\begin{aligned}
&\frac{\zeta_{n+2}^{i^*(m,n+2)} R_{m+i^*(m,n+2)}}{\zeta_{n+1}^{i^*(m,n+1)} R_{m+i^*(m,n+1)}} \\
&= \frac{\zeta_{n+2}^{\frac{R_{m+1}}{R_m}} \cdot \zeta_{n+2}^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_{n+2}^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}} \cdots \zeta_{n+2}^{\frac{R_{m+i^*(m,n+2)}}{R_{m+i^*(m,n+2)-1}}}}{\zeta_{n+1}^{\frac{R_{m+1}}{R_m}} \cdot \zeta_{n+1}^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_{n+1}^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}}} \\
&\leq \frac{\zeta_{n+2}^{\frac{R_{m+1}}{R_m}} \cdot \zeta_{n+2}^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_{n+2}^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}} \cdots \zeta_{n+2}^{\frac{R_{m+i^*(m,n+2)}}{R_{m+i^*(m,n+2)-1}}}}{\zeta_{n+1}^{\frac{R_{m+1}}{R_m}} \cdot \zeta_{n+1}^{\frac{R_{m+2}}{R_{m+1}}} \cdots \zeta_{n+1}^{\frac{R_{m+i^*(m,n+1)}}{R_{m+i^*(m,n+1)-1}}} \cdots \zeta_{n+1}^{\frac{R_{m+i^*(m,n+2)}}{R_{m+i^*(m,n+2)-1}}}} \\
&= \left(\frac{\zeta_{n+2}}{\zeta_{n+1}}\right)^{i^*(m,n+2)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\zeta_{n+1}^r \mathrm{ROTI}_n^{PI} \geq \zeta_{n+2}^r \mathrm{ROTI}_{n+1}^{PI} \\
&\Rightarrow \quad \mathrm{ROTI}_n^{PI} \frac{\zeta_{n+1}^{i^*(m,n+1)} R_{m+i^*(m,n+1)}}{\zeta_n^{i^*(m,n)} R_{m+i^*(m,n)}} \geq \mathrm{ROTI}_{n+1}^{PI} \frac{\zeta_{n+2}^{i^*(m,n+2)} R_{m+i^*(m,n+2)}}{\zeta_{n+1}^{i^*(m,n+1)} R_{m+i^*(m,n+1)}}.
\end{aligned}
$$

■

Proof of Proposition II.3. Because $\delta > \bar{\delta}$, then the expression inside the parenthesis in (II.8) is decreasing by Lemma II.6. Thus,

$$
j^*(m,n) = \max\left\{ j > 0 : \zeta_{n+j} \frac{1-\zeta_{n+j-1}}{1-\zeta_{n+j}} \frac{\zeta_{n+j}^{i^*(m,n+j)} R_{m+i^*(m,n+j)}}{\zeta_{n+j-1}^{i^*(m,n+j-1)} R_{m+i^*(m,n+j-1)}} > 1 \right\},
$$

and the resulting improve-up-to level will be independent of the initial process reliability level q_n. ■

Proof of Proposition II.4. Proposition II.2(i) shows that for any q_n, $q^*(R_m, q_n)$ is nonincreasing in R_m. Moreover, under Assumption II.2 and when $\delta > \bar{\delta}$, this improve-up-to level is independent of initial process reliability level q_n. Therefore, $q^*(R^0_{m_A}) \geq q^*(R^0_{m_B})$.

Since the enhance-up-to rate $R^*(q^*)$ is nondecreasing in q^*, and is independent of the initial revenue rate R^0_m (Proposition II.2), the entrepreneur with the low initial revenue rate will always improve at least up to the same level as the entrepreneur with high initial revenue rate. ■

Proof of Corollaries II.1–II.3. Looking at equation (II.8), if $\text{ROTI}^{PI}_k = \zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}} > 1$, then $j^*(m,n) > 1$ (Corollary II.1). Nevertheless, even if $\zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}} < 1$ it may still be sufficient for $\zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}}\frac{\zeta_{k+1}^{i^*(m,k+1)}R_{m+i^*(m,k+1)}}{\zeta_k^{i^*(m,k)}R_{m+i^*(m,k)}} > 1$ (Corollary II.2). Furthermore, we see that if $\text{ROTI}^{PI}_k = \zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}} > 1\ \forall k$, then the expression $\zeta_{k+1}\frac{1-\zeta_k}{1-\zeta_{k+1}}\frac{\zeta_{k+1}^{i^*(m,k+1)}R_{m+i^*(m,k+1)}}{\zeta_k^{i^*(m,k)}R_{m+i^*(m,k)}} > 1\ \forall k$ (Corollary II.3). ■

Lemma II.7 *Suppose it is optimal to do process improvement in states* $(1, R_m, q_k)$, $\forall k < n$. *Then, engaging in process improvement in state* $(1, R_m, q_n)$ *results in the following relationship,*

$$V(1,R_m,q_n) = \zeta_{\beta,n+1}V(1,R_m,q_{n+1}),$$

$$\textit{where} \qquad \zeta_{\beta,n+1} = \frac{\delta q_{n+1}}{1-\delta(1-q_{n+1})(\beta+(1-\beta)\zeta_{\beta,n})}.$$

Proof of Lemma II.7 We prove by induction.

(Base case) $n = 1$, and assume it is optimal to do PI in state $(1, R_m, q_0)$. With no further process deterioration in process reliability q_0, from expression (II.3) we have.

$$\begin{aligned}
V(0,R_m,q_0) &= 0+\delta q_0 V(1,R_m,q_0)+\delta(1-q_0)V(0,R_m,q_0) \\
&= \zeta_0 V(1,R_m,q_0), \\
V(0,R_m,q_1) &= 0+\beta V(1,R_m,q_0)+(1-\beta)\zeta_0 V(1,R_m,q_0) \\
&= (\beta+(1-\beta)\zeta_0)V(1,R_m,q_0).
\end{aligned}$$

Substituting this expression into the expression for $V(1,R_m,q_0)$, we have

$$\begin{aligned}
V(1,R_m,q_0) &= 0+\delta q_1 V(1,R_m,q_1)+\delta(1-q_1)V(0,R_m,q_1) \\
&= 0+\delta q_1 V(1,R_m,q_1)+\delta(1-q_1)(\beta+(1-\beta)\zeta_0)V(1,R_m,q_0) \\
&= \frac{\delta q_1}{1-\delta(1-q_1)(\beta+(1-\beta)\zeta_0)}V(1,R_m,q_1) \\
&= \zeta_{\beta,1}V(1,R_m,q_1).
\end{aligned}$$

(Induction step) Now suppose that it is optimal to do PI in states $(1,R_m,q_k)$ $\forall k \leq n-2$ and that $V(1,R_m,q_{n-2}) = \zeta_{\beta,n-1}V(1,R_m,q_{n-1})$. Then we have,

$$\begin{aligned}
V(0,R_m,q_n) &= \beta V(1,R_m,q_{n-1})+(1-\beta)V(1,R_m,q_{n-2}) \\
&= (\beta+(1-\beta)\zeta_{\beta,n-1})V(1,R_m,q_{n-1}).
\end{aligned}$$

Substituting this expression into the expression for $V(1,R_m,q_{n-1})$, we have

$$\begin{aligned}
V(1,R_m,q_{n-1}) &= 0+\delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n) \\
&= 0+\delta q_n V(1,R_m,q_n)+\delta(1-q_n)(\beta+(1-\beta)\zeta_{\beta,n-1})V(1,R,q_{n-1}) \\
&= \frac{\delta q_n}{1-\delta(1-q_n)(\beta+(1-\beta)\zeta_{\beta,n-1})}V(1,R,q_n) \\
&= \zeta_{\beta,n}V(1,R,q_n).
\end{aligned}$$

■

Proof of Proposition II.5 The proof of this proposition uses Lemma II.7, which appears in the Appendix. Suppose we do process improvement in states (R_m,q_k) $\forall k \leq n$. Then, from (II.3), we obtain the following relationships:

$$\begin{aligned}
V(0,R_m,q_n) &= \beta V(1,R_m,q_{n-1})+(1-\beta)V(1,R_m,q_{n-2}), \quad \forall n>1, \\
V(0,R_m,q_1) &= \beta V(1,R_m,q_0)+(1-\beta)V(0,R_m,q_0) \\
&= \beta V(1,R_m,q_0)+(1-\beta)\zeta_0 V(1,R_m,q_0).
\end{aligned}$$

Using these equalities, we can rewrite the dynamic program in terms of the non-crisis states as follows,

$$
\begin{aligned}
V(1,R_m,q_n) &= \max\Big\{ \quad 0+\delta q_{n+1}V(1,R_m,q_{n+1})+\delta(1-q_{n+1})V(0,R_m,q_{n+1}), \\
&\qquad 0+\alpha V(0,R_{m+1},q_n)+(1-\alpha)V(0,R_m,q_n), \\
&\qquad R_m+V(0,R_m,q_n) \quad \Big\} \\
&= \max\Big\{ \quad \delta q_{n+1}V(1,R_m,q_{n+1}) \\
&\qquad +\delta(1-q_{n+1})(\beta V(1,R_m,q_n)+(1-\beta)V(1,R_m,q_{n-1})), \\
&\qquad \alpha[\beta V(1,R_{m+1},q_{n-1})+(1-\beta)V(1,R_{m+1},q_{n-2})]+ \\
&\qquad +(1-\alpha)[\beta V(1,R_m,q_{n-1})+(1-\beta)V(1,R_m,q_{n-2})], \\
&\qquad R_m+\beta V(1,R_m,q_{n-1})+(1-\beta)V(1,R_m,q_{n-2}) \quad \Big\} \\
&= \max\Big\{ \quad \delta q_{n+1}V(1,R_m,q_{n+1}) \\
&\qquad +\delta(1-q_{n+1})(\beta+(1-\beta)\zeta_{\beta,n})V(1,R_m,q_n), \\
&\qquad \alpha(\beta+(1-\beta)\zeta_{\beta,n-1})V(1,R_{m+1},q_{n-1}) \\
&\qquad +(1-\alpha)(\beta+(1-\beta)\zeta_{\beta,n-1})V(1,R_m,q_{n-1}), \\
&\qquad R_m+(\beta+(1-\beta)\zeta_{\beta,n-1})V(1,R_m,q_{n-1}) \quad \Big\} \\
&= \max\Big\{ \quad [\delta q_{n+1}+\delta(1-q_{n+1})\times \\
&\qquad\quad (\beta\zeta_{\beta,n+1}+(1-\beta)\zeta_{\beta,n+1}\zeta_{\beta,n})]V(1,R_m,q_{n+1}), \\
&\qquad \alpha(\beta\zeta_{\beta,n}+(1-\beta)\zeta_{\beta,n}\zeta_{\beta,n-1})V(1,R_{m+1},q_n) \\
&\qquad +(1-\alpha)(\beta\zeta_{\beta,n}+(1-\beta)\zeta_{\beta,n}\zeta_{\beta,n-1})V(1,R_m,q_n), \\
&\qquad R_m+(\beta\zeta_{\beta,n}+(1-\beta)\zeta_{\beta,n}\zeta_{\beta,n-1})V(1,R_m,q_n) \quad \Big\} \\
&= \max\Big\{ \quad \zeta_{\beta,n+1}V(1,R_m,q_{n+1}), \\
&\qquad \alpha\zeta_n(\beta)V(1,R_{m+1},q_n)+(1-\alpha)\zeta_n(\beta)V(1,R_m,q_n), \\
&\qquad R_m+\zeta_n(\beta)V(1,R_m,q_n) \quad \Big\},
\end{aligned}
$$

where the third and fourth equality is due to Lemma II.7, and the final equality is because

$$\beta\zeta_{\beta,n} + (1-\beta)\zeta_{\beta,n}\zeta_{\beta,n-1} \equiv \zeta_n(\beta), \quad \text{and}$$

$$\delta(1-q_{n+1})(\beta\zeta_{\beta,n+1} + (1-\beta)\zeta_{\beta,n+1}\zeta_{\beta,n}) + \delta q_{n+1} = \zeta_{\beta,n+1}.$$

Further simplifying the second expression inside the maximization expression as,

$$\begin{aligned} V(1,R_m,q_n) &= \zeta_n(\beta)\alpha V(1,R_{m+1},q_n) + \zeta_n(\beta)(1-\alpha)V(1,R_m,q_n) \\ &= \frac{\zeta_n(\beta)\alpha}{1-\zeta_n(\beta)(1-\alpha)} V(1,R_{m+1},q_n), \end{aligned}$$

we have our expression after dropping the redundant "1",

$$V(R_m,q_n) = \max\{\zeta_{\beta,n+1}V(R_m,q_{n+1}), \zeta_n(\alpha,\beta)V(R_{m+1},q_n), R_m + \zeta_n(\beta)V(R_m,q_n)\}.$$

■

Proof of Lemma II.4 The proof is based on the DP recursion (II.4) and is similar to the proof of Lemma II.2, with appropriate changes in the discount factors. ■

Lemma II.8 *Suppose that the decisions in states $(1,R_m,q_{n-1})$ and $(1,R_{m+1},q_{n-1})$ are RE and RG respectively. Then, denoting $K(m,n) \equiv \delta q_n V(1,R_m,q_n)$ $+\delta(1-q_n)V(0,R_m,q_n)$,*

$$K(m+1,n) - K(m,n) \geq K(m+1,n-1) - K(m,n-1).$$

Proof of Lemma II.8 We have,

$$
\begin{aligned}
K(m+1,n-1) &= \delta q_{n-1}V(1,R_{m+1},q_{n-1})+\delta(1-q_{n-1})V(0,R_{m+1},q_{n-1}) \\
&= \delta q_{n-1}R_{m+1}+\delta q_{n-1}V(0,R_{m+1},q_{n-1}) \\
&\quad +\delta(1-q_{n-1})V(0,R_{m+1},q_{n-1}) \\
&= \delta q_{n-1}R_{m+1}+\delta V(0,R_{m+1},q_{n-1}) \\
&= \delta q_{n-1}R_{m+1}+\delta\beta K(m+1,n-1)+\delta(1-\beta)K(m+1,n-2), \\
K(m,n-1) &= \delta q_{n-1}V(1,R_m,q_{n-1})+\delta(1-q_{n-1})V(0,R_m,q_{n-1}) \\
&= \delta q_{n-1}\alpha V(0,R_{m+1},q_{n-1})+\delta q_{n-1}(1-\alpha)V(0,R_m,q_{n-1}) \\
&\quad +\delta(1-q_{n-1})V(0,R_m,q_{n-1}) \\
&= \delta q_{n-1}\alpha V(0,R_{m+1},q_{n-1})+\delta(1-q_{n-1}\alpha)V(0,R_m,q_{n-1}),
\end{aligned}
$$

$K(m+1,n-1)-K(m,n-1)$

$$
\begin{aligned}
&= \delta q_{n-1}R_{m+1}+\delta(1-q_{n-1}\alpha)V(0,R_{m+1},q_{n-1})-\delta(1-q_{n-1}\alpha)V(0,R_m,q_{n-1}) \\
&= \delta q_{n-1}R_{m+1}+\delta(1-q_{n-1}\alpha)\{\beta K(m+1,n-1)+(1-\beta)K(m+1,n-2)\} \\
&\quad -\delta(1-q_{n-1}\alpha)\{\beta K(m,n-1)+(1-\beta)K(m,n-2)\} \\
&= \delta q_{n-1}R_{m+1}+\delta(1-q_{n-1}\alpha)\beta\{K(m+1,n-1)-K(m,n-1)\} \\
&\quad +\delta(1-q_{n-1}\alpha)(1-\beta)\{K(m+1,n-2)-K(m,n-2)\} \\
&= \frac{\delta q_{n-1}R_{m+1}}{1-\delta(1-q_{n-1}\alpha)\beta}+\frac{\delta(1-\beta)(1-q_{n-1}\alpha)}{1-\delta\beta(1-q_{n-1}\alpha)}\{K(m+1,n-2)-K(m,n-2)\}.
\end{aligned}
$$

We have that

$K(m+1,n-1)-K(m,n-1)$

$$
\begin{aligned}
&= \frac{\delta q_{n-1} R_{m+1}}{1-\delta\beta(1-q_{n-1}\alpha)} + \frac{\delta(1-\beta)(1-q_{n-1}\alpha)}{1-\delta\beta(1-q_{n-1}\alpha)}\{K(m+1,n-2)-K(m,n-2)\} \\
&\leq \frac{\delta q_{n-1} R_{m+1}}{1-\delta(1-q_{n-1}\alpha)} \leq \frac{\delta q_n R_{m+1}}{1-\delta(1-q_n\alpha)},
\end{aligned}
$$

where the first inequality is because the expression is increasing in β since $K(m+1,n-2) \geq K(m,n-2)$ and is therefore bounded from above when $\beta = 1$; and the second inequality is because $\{q_k\}$ is increasing in k. Therefore,

$$
\begin{aligned}
& (1-\delta(1-q_n\alpha))\{K(m+1,n-1)-K(m,n-1)\} \leq \delta q_n R_{m+1} \\
\Leftrightarrow \quad & \frac{1-\delta(1-q_n\alpha)}{1-\delta\beta(1-q_n\alpha)}\{K(m+1,n-1)-K(m,n-1)\} \\
& \qquad \leq \frac{\delta q_n}{1-\delta\beta(1-q_n\alpha)} R_{m+1} \\
\Leftrightarrow \quad & \left\{1-\frac{\delta(1-\beta)(1-q_n\alpha)}{1-\delta\beta(1-q_n\alpha)}\right\}\{K(m+1,n-1)-K(m,n-1)\} \\
& \qquad \leq \frac{\delta q_n}{1-\delta\beta(1-q_n\alpha)} R_{m+1} \\
\Leftrightarrow \quad & \{K(m+1,n-1)-K(m,n-1)\} \leq \frac{\delta q_n R_{m+1}}{1-\delta\beta(1-q_n\alpha)} \\
& \qquad + \frac{\delta(1-\beta)(1-q_n\alpha)}{1-\delta\beta(1-q_n\alpha)}\{K(m+1,n-1)-K(m,n-1)\} \\
\Leftrightarrow \quad & \{K(m+1,n-1)-K(m,n-1)\} \leq \{K(m+1,n)-K(m,n)\}.
\end{aligned}
$$

■

Proof of Proposition II.6 We prove by construction.

(i) For any $m < M$, suppose that in state $(1, R_{m+1}, q_k)$ $\forall q_k \leq \bar{q}$, it is optimal to do process improvement up to $q^*(R_{m+1}) > \bar{q}$. Then, starting in state $(1, R_m, q_k)$ $\forall q_k \leq \bar{q}$, by Lemma II.4, it is suboptimal to do revenue enhancement for all process reliability $q < q^*(R_{m+1})$. Thus $q^*(R_m) \geq q^*(R_{m+1})$.

(ii) Suppose that the optimal decisions in states $(1, R_m, q_n)$ and $(1, R_{m+1}, q_n)$ are RE and RG respectively. To show that the enhance-up-to level is nondecreasing in n, we will show that RE dominates RG in state $(1, R_m, q_{n+1})$. In state $(1, R_m, q_n)$, from expression (II.3), we have

$$\begin{aligned}
\alpha V(0,R_{m+1},q_n)+(1-\alpha)V(0,R_m,q_n) &> R_m+V(0,R_m,q_n) \qquad \text{(II.15)}\\
\Leftrightarrow \quad V(0,R_{m+1},q_n) &> \frac{R_m}{\alpha}+V(0,R_m,q_n)\\
\Leftrightarrow \quad V(0,R_{m+1},q_n) &> \{R_m+V(0,R_m,q_n)\}+\frac{1-\alpha}{\alpha}R_m,
\end{aligned}$$

Furthermore, defining $K(m,n) \equiv \delta q_n V(1,R_m,q_n)+\delta(1-q_n)V(0,R_m,q_n)$ (Lemma II.8), and $\zeta_n^\beta \equiv \frac{\delta\beta q_n}{1-\delta\beta(1-q_n)}$, we have

$$\begin{aligned}
V(0,R_{m+1},q_n) &= 0+\beta\Big[\delta q_n V(1,R_{m+1},q_n)+\delta(1-q_n)V(0,R_{m+1},q_n)\Big]+\\
&\quad +(1-\beta)\Big[\delta q_{n-1}V(1,R_{m+1},q_{n-1})+\delta(1-q_{n-1})V(0,R_{m+1},q_{n-1})\Big]\\
&= 0+\beta\Big[\delta q_n V(1,R_{m+1},q_n)+\delta(1-q_n)V(0,R_{m+1},q_n)\Big]\\
&\quad +(1-\beta)K(m+1,n-1)\\
&= \frac{\delta\beta q_n}{1-\delta\beta(1-q_n)}V(1,R_{m+1},q_n)+\frac{(1-\beta)}{1-\beta\delta(1-q_n)}K(m+1,n-1),\\
&= \zeta_n^\beta(R_{m+1}+V(0,R_{m+1},q_n))+\frac{(1-\beta)}{1-\beta\delta(1-q_n)}K(m+1,n-1),
\end{aligned}$$

in which the final equality is because it is optimal to do RG in state $(1,R_{m+1},q_n)$ by assumption. Solving for $V(0,R_{m+1},q_n)$, we have

$$\begin{aligned}
V(0,R_{m+1},q_n) &= \frac{\zeta_n^\beta}{1-\zeta_n^\beta}R_{m+1}+\frac{1}{1-\zeta_n^\beta}\frac{1-\beta}{1-\beta\delta(1-q_n)}K(m+1,n-1)\\
&= \frac{\zeta_n^\beta}{1-\zeta_n^\beta}R_{m+1}+\left(\frac{1-\beta}{1-\beta\delta}\right)K(m+1,n-1). \qquad \text{(II.16)}
\end{aligned}$$

Using a similar logic, we can show that, because RG is not necessarily optimal in state $(1,R_m,q_n)$,

$$V(0,R_m,q_n) \geq \frac{\zeta_n^\beta}{1-\zeta_n^\beta}R_m+\left(\frac{1-\beta}{1-\beta\delta}\right)K(m,n-1). \qquad \text{(II.17)}$$

Substituting the expressions (II.16) and (II.17) back into the inequality (II.15), we have

$$\frac{\zeta_n^\beta}{1-\zeta_n^\beta}R_{m+1}+\left(\frac{1-\beta}{1-\beta\delta}\right)K(m+1,n-1)>\frac{R_m}{1-\zeta_n^\beta} \qquad \text{(II.18)}$$

$$+\left(\frac{1-\beta}{1-\beta\delta}\right)K(m,n-1)+\left(\frac{1-\alpha}{\alpha}\right)R_m$$

$$\Leftrightarrow \quad \frac{\zeta_n^\beta}{1-\zeta_n^\beta}R_{m+1}-\frac{R_m}{1-\zeta_n^\beta} \qquad \text{(II.19)}$$

$$>\left(\frac{1-\beta}{1-\beta\delta}\right)\{K(m,n-1)-K(m+1,n-1)\}+\left(\frac{1-\alpha}{\alpha}\right)R_m$$

$$\Leftrightarrow \quad \frac{\beta\delta}{1-\beta\delta}q_n(R_{m+1}-R_m)-\frac{R_m}{\alpha}$$

$$>\left(\frac{1-\beta}{1-\beta\delta}\right)\{K(m,n-1)-K(m+1,n-1)\}. \qquad \text{(II.20)}$$

We have

$$\begin{aligned}\frac{\beta\delta}{1-\beta\delta}q_{n+1}(R_{m+1}-R_m)-\frac{R_m}{\alpha} &> \frac{\beta\delta}{1-\beta\delta}q_n(R_{m+1}-R_m)-\frac{R_m}{\alpha}\\ &> \left(\frac{1-\beta}{1-\beta\delta}\right)\{K(m,n-1)-K(m+1,n-1)\}\\ &> \left(\frac{1-\beta}{1-\beta\delta}\right)\{K^{RE}(m,n-1)-K(m+1,n-1)\}\\ &> \left(\frac{1-\beta}{1-\beta\delta}\right)\{K(m,n)-K(m+1,n)\},\end{aligned}$$

where the first inequality is because $\{q_k\}$ is increasing in k, the second due to equation (II.20), the third because in $K^{RE}(m,n-1)=\delta q_{n-1}V^{RE}(1,R_m,q_{n-1})$ $+\delta(1-q_{n-1})V^{RE}(0,R_m,q_{n-1})$ we assume that the decision in state $(1,R_m,q_{n-1})$ is RE, which may be suboptimal, and the final inequality is due to Lemma II.8.

Rolling back, we have that in state $(1,R_m,q_{n+1})$, doing RG is dominated by doing RE followed by doing RG, i.e.

$$V^{RG}(0,R_{m+1},q_{n+1})>\frac{R_m}{\alpha}+V(0,R_m,q_{n+1}).$$

Since doing RG in state $(1, R_{m+1}, q_{n+1})$ is not necessarily the optimal decision, we have

$$V(0, R_{m+1}, q_{n+1}) \geq V^{RG}(0, R_{m+1}, q_{n+1}) > \frac{R_m}{\alpha} + V(0, R_m, q_{n+1}).$$

■

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CHAPTER III

The Time-Money Tradeoff in the Entrepreneurial Production and Hiring

The supply of time is totally inelastic. No matter how high the demand, the supply will not go up. There is no price for it. Time is totally perishable and cannot be stored. Yesterday's time is gone forever, and will never come back. Time is always in short supply.

– Peter F. Drucker

1 Introduction

Consider an entrepreneurial firm during its growth phase, when the entrepreneur is the primary decision maker. As the firm expands, an increasing number of tasks (of varying importance) surfaces which require the entrepreneur's attention, creating an overwhelming demand on the entrepreneur's time (Gifford 1992). Because the supply of time remains constant, to sustain growth, the entrepreneurs must hire employees and delegate the low value-adding tasks, to free up time for the high value-adding activities in return for the wage payments.

Simultaneously however, the entrepreneurs are constantly cash-starved as there exists an extraordinary need for additional funds to keep up with the pace of growth

(Hambrick and Crozier 1985), often leaving them resorting to financial bootstrapping to fund growth (Ebben and Johnson 2006). Thus, whether or not the value of additional time gained would offset the cost of wage payments is unclear. Furthermore, for firms without the managerial resources for systematic recruiting efforts (Aldrich and Fiol 1994) or the legitimacy to attract highly qualified employees (Williamson 2000), the process of hiring entails significant upfront investments of both entrepreneurs' time and money and thus may interfere with revenue production and growth (Cook 1999).

What is the relative prices of additional time and additional money? When is it appropriate for the entrepreneur to seek a new employee? How should the associated setup time, setup cost, and wage influence the timing of the hiring decision, and how does the optimal timing affect the shadow values of time and money? While it is clear that hiring is critical for sustained growth of organizations (e.g. Koch and McGrath 1996), there exist to date few explicit guidelines for the timing of hiring decision faced by entrepreneurial firms.

In this paper, we present a formalized model of the entrepreneurial production and outline how the inputs of time and money interact over time. Motivated by the theory of constraints (Goldratt and Cox 2004), we model the entrepreneur's production as a function of time and money, the chief constraints of entrepreneurial firms during growth (Klaas et al. 2000, Hambrick and Crozier 1985). Specifically, as both time and money are necessary for generating revenue, we employ the Cobb Douglas function, representing money and time as complementary resources. Viewing hiring as a tradeoff between the fungible resources of time and money (e.g. Soman 2001, Devoe and Pfeffer 2007), we characterize the optimal timing of the hiring decision faced by the entrepreneurial firms.

We demonstrate that the shadow value of time always becomes greater than the shadow value of money, ultimately making time the key bottleneck resource. Moreover, we establish that there exists a unique cash level threshold above which it is optimal to hire. In fact, we find that the optimal timing of hiring maximizes the post-hire gap between the shadow values of time and money. Furthermore, we find that the hiring threshold is nonmonotonic in the setup time associated with hiring, due to the tradeoff between the need to delay hiring to preserve the growth momentum and the need to expedite hiring given that the shadow value of time is increasing. By contrast, the threshold is always increasing in the setup cost, thus highlighting the importance of differentiating setup cost with setup time. We test the robustness of our model to our assumptions and present further insights by generalizing the production function to the constant-elasticity-of-substitution (CES) production function and extending our results to multiple hiring decisions.

The paper is organized as follows. We review the related literature in the next section. Section 3 introduces the basic model and characterizes its optimal solutions, in particular the evolution of the shadow values of time and money during the firm's growth. We introduce in §4 the hiring decision of the entrepreneur as a way to trade off money against time, characterize the optimal hiring policy, and derive the comparative statics with respect to the hiring setup cost and setup time. Section 5 tests the robustness of our model to our assumptions and presents further insights. We present our conclusions and directions for future research in §6. All proofs appear in the Appendix.

2 Literature Review

In this section, we first examine the research on organizational life-cycle of firms and motivate time and money as the key constrained inputs to the entrepreneurial production function during the growth phase. Then, we review the related literature on hiring decisions.

2.1 The Entrepreneurial Production Function

We first formalize the growth phase of the entrepreneurial firm in the context of the organizational life cycle research. We will then review the literature which find entrepreneur's time and money as the key constrained resources during high growth.

2.1.1 Phases in the Organizational Life Cycles.

A myriad of research on organizational life cycles examines the organizational evolution of firms from their birth to maturity. In particular, Quinn and Cameron (1983), in a review of nine different life cycle models, notes that most models contain the following four stages: (1) entrepreneurial stage (early innovation, niche formation, creativity); (2) collectivity stage (high cohesion, commitment); (3) formalized and control stage (stability and institutionalization); and (4) structure elaboration and adaptation stage (domain expansion and decentralization). While the length of early stage is firm-dependent, they note that "a consistent pattern of development seem to occur in organizations over time, and organizational activities and structure in one stage are not the same as the activities and structure present in another stage."

In contrast to firms in the first stage, whose primary objective is survival (Steinmetz

1969), in the second stage, firms are considered to have established a market niche with their product or service, i.e. have passed the "survival threshold", and focus on sustaining their growth. During this phase, while the entrepreneur maintains direct control of major activities, informal structures and communication mechanisms develop. The third stage and fourth stage describes the maturation phases where the firms shed their entrepreneurial characteristics and transforms into a professional organization.

While most research on entrepreneurial firms focuses on the first stage of the organizational life-cycle – i.e. discovery of entrepreneurial opportunities, the process of innovation, decisions to maximize the survival likelihood (e.g. Shane and Venkataraman 2000), we focus on entrepreneurial firms in the second stage. In particular, we focus on the entrepreneurs who seek cash-out opportunities and whose objective is to maximize the value of the firm prior to sale (e.g. Bygrave and Zacharakis 2010, Babich and Sobel 2004).

2.1.2 Time and Money as Inputs.

During the growth phase, entrepreneurs are simultaneously limited in both time and money. The constraints on time during growth is primarily caused by the increasing demands for the entrepreneur's attention. Simon (1976, pg. 294) argues that "attention is the chief bottleneck in organizational activity, and the bottleneck becomes narrower and narrower as we move to the tops of organizations." Gifford (1992) argues that the attention bottleneck is particularly pronounced and sustains throughout the high-growth phase because the demand for the entrepreneur's attention increases due to the tasks endogenously created by the entrepreneur's allocation of attention to the growth activities. Nonetheless, during this growth phase, entrepreneurs' attention should not be spared as "a CEO who knows everything that is going on and pays attention to the smallest details . . . [is] beneficial and necessary for the company" (Flamholtz 1986). As such, various time allocation models in the entrepreneurial context have been examined (e.g. Levesque and MacCrimmon 1998, Levesque et al. 2002, Levesque et al. 2005).

Moreover, during the growth phase, in order to keep up with the pace of growth, entrepreneurs are constantly cash-starved. While the initial financial capital is important for survival and achieving high growth (Cooper et al. 1994), there is an ongoing need for additional cash for new machinery, equipment, talent, etc., to fuel growth under changing market conditions (Hambrick and Crozier 1985). As a result, entrepreneurs constantly search for additional funds, either externally or internally via financial bootstrapping (Ebben and Johnson 2006). As such, the cash limitations has been the basis for many entrepreneurial operations management models (e.g. Swinney et al. 2005, Archibald et al. 2002).

In this paper, applying the prescriptions of the theory of constraints (Goldratt and Cox 2004), we model the entrepreneur's production as a function of time and money, the chief constraints of entrepreneurial firms during growth. Although entrepreneurial production models incorporating labor and capital have been introduced (e.g. Garmaise 2008), the study of dynamic evolution of the constraints or the exchange between the two resources have been ignored. Because time and money are fungible resources (e.g. Soman 2001, LeClerc et al. 1995, Devoe and Pfeffer 2007, Okada and Hoch 2004) where time (money) can often be replaced by money (time) during production, we employ the Cobb-Douglas production function – with time and money as the complementary inputs – to examine how to trade off money against time.

2.2 Hiring Decisions

We review two strands of literature related to the timing of hiring decisions faced by resource constrained entrepreneurial firms and highlight our contributions. The first literature provides *descriptive* insights of human resource management decisions, whereas the second strand presents *prescriptive* insights to hiring.

2.2.1 Empirical Research.

For firms operating in complex and dynamic competitive environments, their human capital is an important source of sustained competitive advantage (Hitt et al. 2001). For example, Koch and McGrath (1996) empirically show that firms utilizing more sophisticated human resource planning, recruitment, and selection strategies show higher labor productivity, whereas Kor (2003) reports that diversity of managerial experiences in the top management team allows firms to better seize new growth opportunities. The human resources management capability becomes particularly important for small firms during rapid growth phases (Thakur 1998), as there exist significant recruiting and training needs as job demands expand continually (Kotter and Sathe 1978), and because they must make productive use of their limited resources (Siegel et al. 1993).

The process of hiring, however, presents a significant challenge for growth-oriented firms constrained in time and money (Tansky and Heneman 2006, Klaas et al. 2000). For example, there is an extraordinary need for additional funds to keep up with the pace of growth, even when the firm is profitable (Hambrick and Crozier 1985), and the time consuming nature of many complex human resources (HR) activities interfere with managerial responsibilities that are directly related to revenue production (Cook 1999). The resource drain associated with hiring often arises when the firms lack the legitimacy to recruit the skills they need (Williamson 2000), or when formal HR systems do not exist (Aldrich and Fiol 1994).

When hiring takes place, tasks are often created after new employees are hired rather than employees being hired to perform specific tasks, due to the lack of established departmental boundaries or standardized tasks or roles (Flamholtz 1986). In such *opportunistic hiring* scenarios, unlike the task-oriented hiring scenarios where the employee's specific skills are sought after, the organizational fit of the employee becomes critical (Levesque 2005). Accordingly, it is found that the top manager's social networks replaces the core HR functions associated with hiring (Collins and Clarck 2003).

In contrast to these empirical descriptions of the hiring decisions faced by entrepreneurial firms, we develop a formal model of hiring for entrepreneurial firms and characterize the optimal timing of the hiring decisions.

2.2.2 Prescriptive Research.

Although staffing has been widely studied in the operations management literature, it has been mostly studied in the context of established firms with existing demands and sufficient resources.

Optimal hiring policies have been derived to cope with intra or inter-day variations, respectively using queuing or mathematical programming approaches. Using a queuing model, Bassamboo et al. (2006) propose call center staffing policies that minimize the sum of personnel costs and abandonment penalties, while Pinker and Shumsky (2000) and Gans and Zhou (2002) refine the queuing model to account for specialization and learning.

In contrast to queuing models, which consider a stationary but stochastic environment to study intra-day variations, mathematical programming approaches typically consider a nonstationary, but deterministic environment to study inter-day variations. In particular, Holt et al. (1960) examine in their seminal paper, the workforce levels to minimize the long-run costs of overtime/idle time, and hiring/layoff costs. Production smoothing models have evolved to derive the optimal hiring and promotion policies by considering differences in productivity between experienced workers and new hires (Orrbeck et al. 1968), learning curves (Ebert 1976), organizational age of workers and their effectiveness (Gaimon and Thompson 1984), and stochastic employee turnovers (e.g. Bordoloi and Matsuo 2002). Under similar settings, Aksin (2007) develops a modeling framework to assesses the marginal value of a human asset. However, whereas in these studies the demand is typically considered exogenous, we consider an endogenous demand (through the Cobb-Douglas production function), as entrepreneurs often create their own demand (Schumpeter 1934).

3 Basic Model

In this section, we introduce the assumptions, present the model, and discuss the optimal solution.

3.1 Assumptions

We consider a dynamic model where the entrepreneurial firm accumulates revenue over time. We assume that entrepreneurs endogenously create demand for their product or services (Schumpeter 1934), and thus characterize the firm's revenue in each period t as a function of the entrepreneur's time investment (T_t) and money investment

(M_t), which represent the two most constrained resources of an entrepreneurial firm (Tansky and Heneman 2006). Moreover, we require that both time and money are required to produce revenue (i.e. complementary), and we assume constant elasticity of substitution between the two resources.

Our model focuses on the entrepreneurial firms belonging to the second stage in the organization life cycle model (according to summary model by Quinn and Cameron 1983), where the firm is considered to have established a market niche with their new product or service, and therefore have passed the "survival threshold" (Steinmetz 1969). This phase characterizes the phase where the main objective is to capitalize on the growth opportunities and to maximize the value of the firm prior to sale. Therefore, we assume that a lack of cash would constrain the rate of growth but would not make the firm at the risk of bankruptcy.

Moreover, during this phase, informal structures and communication mechanisms develop and the entrepreneur's focus shifts from the intra-day variations to inter-day growth trajectories (McCarthy et al. 1990). Accordingly, we consider each time period to be long enough (e.g. month) such that the temporal aggregation marginalizes the day-to-day variability in revenue, and assume that the firm's revenue each period is deterministic, similar to the production smoothing models (Holt et al. 1960). In particular, we employ the Cobb-Douglas (C-D) production function (we generalize to the CES production function of Solow (1956) in §5.1) to model the revenue R_t and profit Π_t earned in each period t as follows:

$$R(M_t, T_t) = KM_t^{\alpha}T_t^{\beta}, \qquad \Pi(M_t, T_t) = KM_t^{\alpha}T_t^{\beta} - M_t.$$

The coefficient K represents the entrepreneur's productivity (or entrepreneur's individual talent) inherent to the firm. The parameters α and β represent the revenue's sensitivity on money and time investments respectively, and can be also interpreted as the inherent characteristics of the industry the entrepreneurs operate in (e.g. Arrow et al. 1961). We assume these parameters do not change over time. Furthermore, we impose the following assumption on the parameters.

Assumption III.1 *(i) K is s.t. $\{M|KM^{\alpha}T^{\beta} > M\} \neq \emptyset$, (ii) $\alpha < 1$.*

The first assumption assures that the parameter K is large enough that investing money in the business is profitable; the second assumption states that the money investment will see a decreasing marginal return on the revenue. Note that despite Assumption III.1(ii), the firm's returns-to-scale can be decreasing ($\alpha+\beta < 1$), constant ($\alpha+\beta = 1$), or increasing ($\alpha+\beta > 1$).

3.2 Model

In the beginning of each period t, the state of the firm is characterized by the available money I_t and the available time J_t. The initial constraint on money I_0 represents the seed capital raised to start the business (i.e. from credit cards, personal savings, friends and family, or angel investors). Because time cannot be inventoried, the available time each period is assumed to remain constant, i.e. $J_t = J\ \forall t$. The money not invested into the firm in period t is available in period $t+1$ after earning an interest of $r > 0$, whereas the time not invested in period t will not carry over to period $t+1$. As a function of the interest rate, we express the discount factor as $\delta = \frac{1}{1+r} < 1$.

We note the flexibility δ provides in the interpretation and modeling. First, the $1-\delta$ can be interpreted as the probability that firm goes bankrupt in the next period, i.e. a low δ implies a high risk business. This is similar to the risk-adjusted net-present-value (or rNPV) framework used in finance to value risky future cash flows. Secondly, the discount factor δ allows for the growth rate of the discounted cash position to slow over time, characterizing the decreasing growth rate of firms in practice (Penrose 1959).

The money inventory dynamics is as follows. Suppose the entrepreneur invests money $M_t \leq I_t$ and time $T_t \leq J$ in period t. Then the money invested into the business will become $KM_t^{\alpha}T_t^{\beta}$ in period $t+1$, while the uninvested money $I_t - M_t$ will gain interest to become $(1+r)(I_t - M_t)$ in period $t+1$. Thus we have:

$$I_{t+1} = (1+r)(I_t - M_t) + KM_t^{\alpha}T_t^{\beta}.$$

We assume the entrepreneur's objective is to maximize the value of the firm when they offer for sale (Bygrave and Zacharakis 2010) or IPO (Babich and Sobel 2004). We model this by the entrepreneur's discounted cash position in period N, i.e. $\delta^{N-1}I_N$. Thus, we have the following model:

$$\begin{array}{lll} \max_{\{M_t,T_t\}} & \delta^{N-1}I_N(M_0,\ldots,M_{N-1};T_0,\ldots,T_{N-1}) & \\ \text{s.t.} & M_t \leq I_t, \quad T_t \leq J_t, & t=0,\ldots,N-1, \\ & I_{t+1} = (1+r)(I_t - M_t) + KM_t^{\alpha}T_t^{\beta}, & t=0,\ldots,N-1, \\ & M_t, T_t \geq 0, & t=0,\ldots,N-1, \end{array} \qquad \text{(III.1)}$$

After recursively rearranging the terms, we have

$$\begin{aligned} I_{t+1} &= (1+r)^{t+1}I_0 + \sum_{k=0}^{t}(1+r)^{t-k}\left(\delta KM_k^{\alpha}T_k^{\beta} - M_k\right) \\ &= (1+r)^{t+1}I_0 + \sum_{k=0}^{t}(1+r)^{t-k}\Pi_k(M_k,T_k). \end{aligned}$$

After substituting the above expression in the constraint and in the objective, we can convert (III.1) to:

$$\max_{\{M_t,T_t\}} \quad \sum_{t=0}^{N-1} \delta^t \Pi_t(M_t,T_t) \tag{III.2}$$

$$\text{s.t.} \quad M_t \leq (1+r)^t I_0 + \sum_{k=0}^{t-1} (1+r)^{t-k} \Pi_k(M_k,T_k), \qquad t=0,\ldots,N-1, \tag{III.3}$$

$$T_t \leq J_t, \qquad t=0,\ldots,N-1, \tag{III.4}$$

$$M_t, T_t \geq 0, \qquad t=0,\ldots,N-1. \tag{III.5}$$

In other words, the objective of maximizing the final period's discounted ash position can be equivalently thought of as maximizing the discounted sum of profits. Since $\Pi_t(M_t,T_t) \leq \Pi_t(M_t,J) < \infty$, the objective $\delta^{N-1} I_N$ converges to a finite value as $N \longrightarrow \infty$ (Bertsekas 2000), making the model extendable to an infinite horizon.

3.3 Optimal Solution

In this section, we analyze the optimal investment M_t^* and T_t^* for each t, and show the evolution of the shadow values of money and time, μ_t^* and τ_t^* associated with constraints (III.3) and (III.4), revealing a bottleneck shift over time. The following proposition presents the optimal investments and their respective shadow values.

Proposition III.1 *The optimal investment of money and time each period, M_t^* and T_t^* respectively, and their respective shadow values μ_t^* and τ_t^* are given by the following expressions:*

$$M_t^* = \min\left\{ I_t, \left(\delta\alpha K J^\beta\right)^{\frac{1}{1-\alpha}} \right\}, \qquad T_t^* = J;$$

$$\mu_t^* = \delta^{t-1} \left(\prod_{k=t+1}^{N} \frac{\alpha K (M_k^*)^\alpha (T_k^*)^\beta}{M_k^*} \right) \left(\frac{\alpha K (M_t^*)^\alpha (T_t^*)^\beta}{M_t^*} - 1 \right),$$

$$\tau_t^* = \delta^{t-1} \left(\prod_{k=t+1}^{N} \frac{\alpha K (M_k^*)^\alpha (T_k^*)^\beta}{M_k^*} \right) \left(\frac{\beta K (M_t^*)^\alpha (T_t^*)^\beta}{T_t^*} \right).$$

Following the optimal policy, during the periods t for which $I_t \leq \left(\delta\alpha K J^\beta\right)^{\frac{1}{1-\alpha}}$, the per-period profit Π_t is increasing in t, and the cash position grows at an increasing rate. When $I_t > \left(\delta\alpha K J^\beta\right)^{\frac{1}{1-\alpha}}$, $M_t^* = \left(\delta\alpha K J^\beta\right)^{\frac{1}{1-\alpha}}$, and the per-period profit Π_t remains constant at $K(M_t^*)^\alpha J^\beta - M_t^*$, thus resulting in the cash position to grow at a constant rate each period.

The term $\prod_{k=t+1}^{N} \frac{\alpha K (M_k^*)^\alpha (T_k^*)^\beta}{M_k^*}$, which appears in both μ_t and τ_t denotes the *compounding effect*, as it describes the influence in the future revenues from periods $t+1$ to N. Because $\frac{\alpha K (M_k^*)^\alpha (T_k^*)^\beta}{M_k^*} \geq 1$, this term is increasing in the remaining periods-to-go $(N-t)$. On the other hand, the coefficients $\frac{\alpha K (M_t^*)^\alpha (T_t^*)^\beta}{M_t^*} - 1$ and $\frac{\beta K (M_t^*)^\alpha (T_t^*)^\beta}{T_t^*}$ respectively denote the *current effect*, or the immediate impact of additional money and time on the current period profit.

The following lemma formalizes the time period in which it is optimal to invest all the earned revenue back into the business.

Lemma III.1 *Let I_0 be the initial seed capital, and let*

$$N^* = \left\lceil \log\left(\frac{\log\left(\frac{1}{\alpha}\right)}{\log\left(\frac{\delta K I_0^\alpha J^\beta}{I_0}\right)}\right)\frac{1}{\log\alpha}\right\rceil.$$

Then, it is optimal to invest the entire revenue back (i.e. $M_t^* = I_t$) into the business if and only if $t \leq N^*$.

We will refer to the period $\{t : t \leq N^*\}$ as the *bootstrapping phase*, as it is commonly known in practice (Ebben and Johnson 2006). Note that the length of the bootstrapping phase decreases in I_0, and increases in α, β, δ, K, and J. In particular, as $\alpha \to 1$, $N^* \to \infty$. In other words, the bootstrapping phase, during which additional money can increase the growth rate, can be arbitrarily long. This is consistent with the common belief that cash is the most constrained resource.

The next corollary formalizes the meaning of the shadow values $\{\mu_t^*\}$ and $\{\tau_t^*\}$.

Corollary III.1

$$\mu_t^* = \frac{\partial}{\partial I_t}\left(\sum_{t=0}^{N-1} \delta^t \Pi_t(M_t, T_t)\right), \qquad \tau_t^* = \frac{\partial}{\partial J_t}\left(\sum_{t=0}^{N-1} \delta^t \Pi_t(M_t, T_t)\right).$$

In other words, the shadow values $\{\mu_t^*\}$ and $\{\tau_t^*\}$ respectively denote the incremental increase in the objective function $\delta^{N-1} I_N(M_0, \ldots, M_t, \ldots, M_{N-1}; T_0, \ldots, T_t, \ldots, T_{N-1})$ when the constraint placed on the available money I_t and the constraint place on the available time J_t are incrementally relaxed. In other words, if $M_t \leq I_t + u$, $T_t \leq J + v$, for small u, v,

$$I_N(M_0, \cdots, M_t + u, \cdots, M_{N-1}; T_0, \cdots, T_t + v, \cdots, T_{N-1})$$
$$= I_N(M_0, \cdots, M_{N-1}; T_0, \cdots, T_{N-1}) + \mu_t^* u + \tau_t^* v.$$

We illustrate the properties of the shadow values μ_t^* and τ_t^* next.

Theorem III.1

i) The shadow value of money μ_t^* is nonincreasing in t for any $\delta < 1$.

ii) For $t < N^* - 1$, the shadow value on time τ_t^* is exponentially increasing in t; and for $t > N^*$, the shadow value of time τ_t^* is exponentially decreasing in t.

iii) For $t < N^* - 1$, the difference between the shadow value of time and the shadow value of money, $\tau_t^* - \mu_t^*$, is increasing in t. Moreover, for $t \geq N^*$, $\tau_t^* > \mu_t^*$.

The theorem states that money becomes a less valuable resource during the bootstrapping phase whereas time becomes a more valuable resource. In particular, Theorem III.1 becomes more pronounced as $\delta \to 1$, where (i) μ_t^* is decreasing for $t < N^*$ and $\mu^* = 0$ for $t \geq N^*$, (ii) τ_t is exponentially increasing for $t < N^*$, and $\tau_t = \tau_{N^*}$ for all $t \geq N^*$, and (iii) $\tau_t^* - \mu_t^*$ is increasing in t. An example of the evolution of the shadow values is illustrated in Figure III.1.

This dynamic evolution of the shadow prices is due to two factors: the physical difference between time and money, and their complementary nature in the production function. First, because the revenue earned in period t can be reinvested into the firm as money in period $t+1$, the available money each period increases, whereas the available time, which cannot be stored, remains constant each period. Therefore, time becomes relatively more scarce than money in subsequent periods. Secondly, because both time and money are required to generate revenue (complementary resources), the relative value of time is greater when there is more available money to invest. Therefore the value of additional time increases as the supply of money increases.

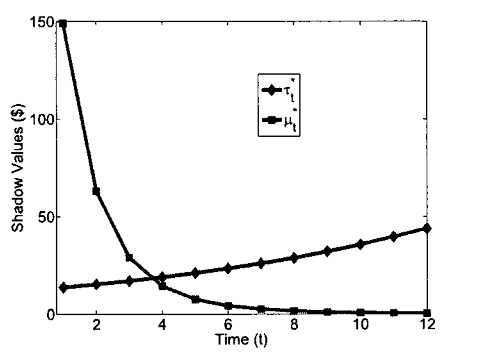

Figure III.1: Example of shadow values. ($\alpha = 0.9, \beta = 0.3, K = 1.1, I_1 = 3, J = 20$, $\delta = 0.95$)

Figure III.1 illustrates that in the early phases of growth, the main bottleneck resource of entrepreneurial firms is cash, consistent with the findings that early financial investments matter in achieving high growth (Cooper et al. 1994) and supports the assumptions of earlier studies on entrepreneurial operations (e.g. Archibald et al. 2002, Swinney et al. 2005). However, after establishing a healthy revenue stream, the main bottleneck resource becomes time. The next corollary gives insights to when the main bottleneck resource switches from money to time.

Corollary III.2 $\tau^* > \mu^*$ *if and only if* $I > \tilde{I}$*, where* $\tilde{I}$ *is the unique cash position satisfying the expression*

$$\beta\tilde{I} + \left(\frac{J^{1-\beta}}{K}\right)\tilde{I}^{1-\alpha} = \alpha J.$$

We note that if the initial seed capital I_0 is large enough (i.e. $I_0 > \tilde{I}$), the main bottleneck resource will always be time. Moreover, using the implicit function theorem, we observe that the bottleneck shift occurs later if K increases ($\frac{\partial \tilde{I}}{\partial K} > 0$), and earlier if J increases ($\frac{\partial \tilde{I}}{\partial J} > 0$) when the time investment has decreasing marginal return on the revenue ($\beta < 1$); however, monotonicity is not necessarily guaranteed for other parameters (We have $\frac{\partial \tilde{I}}{\partial J} > 0 \Leftrightarrow \beta < \alpha K\tilde{I}^{\alpha-1}J^{\beta} + 1$; $\frac{\partial \tilde{I}}{\partial \alpha} > 0 \Leftrightarrow \tilde{I} > 1$; and $\frac{\partial \tilde{I}}{\partial \beta} < 0 \Leftrightarrow KI^{\alpha}J^{\beta} > J\ln J$). We test the robustness of our results to the functional form of the production function in §5.1.

4 Hiring Decision

The previous section showed that the value of additional time eventually becomes greater than the value of addition money, creating a fundamental gap. In this section, we investigate how to lower this disparity by exchanging money against time. Specifically, when the key constraint is time, the entrepreneur can create more time by hiring, thus relieving the bottleneck constraint and increasing profits. We characterize when the entrepreneur should hire their first employee and study the sensitivity of this optimal decision to the hiring setup time and setup cost. We assume that the entrepreneur can hire only one employee, but relax this assumption in §5.2.

4.1 Hiring Decision Framework

For each subsequent period (e.g. month) after hiring, the entrepreneur gains additional time in return for paying a wage to the employee. In particular, due to the difference in the efficiency between the entrepreneur and the employee in completing tasks, the additional time the employee provides is worth only as much as the time the entrepreneur would have needed to spend. Thus, we refer to the time spent by the employee in terms of that converted into equivalent entrepreneur's time units. We denote the wage (variable cost) as w, and the additional time in the entrepreneur's time units as y.

For entrepreneurial firms, moreover, hiring is a complex and time-consuming task which can pose a significant drain on the existing resources (Klaas et al. 2000), and interfere with managerial responsibilities that are directly related to revenue production (Cook 1999). For example, carefully screening applications and assessing the fit and values of candidates and training them require significant upfront investment in time, whereas significant upfront cost is incurred to advertise the position (or hire a head hunter), accommodate additional office space, computers, and other infrastructures. We denote the associated setup cost as $S_M \geq 0$, and the associated setup time as $S_T \geq 0$. The firm incurs setup cost and setup time only during the hiring period, and the exchange between time and money comes into effect each period only after the hiring period.

Let the available money and available time in the beginning of period s be denoted I_s and $J_s = J$ respectively, and suppose the entrepreneur decides to hire in period s. Then, due to the setup cost and setup time incurred, the available money and available time to invest in revenue generating activity will decrease from I_s to $I_s - S_M$ and from J to $J - S_T$ respectively. Therefore, the revenue earned in period s when hiring takes place is no greater than the revenue earned if hiring did not take place. In the subsequent periods t following the hiring period (i.e. $t \in \{s+1,\ldots,N\}$), the entrepreneur's available time increases from J to $J+y$, and before each period ends, the entrepreneur pays the employee a constant wage of w, reducing the available capital in the beginning of period $t+1$ from I_{t+1} to $I_{t+1} - w$.

The following corollary modifies the optimal allocations $\{T_t^*\}$ and $\{M_t^*\}$ and the resulting shadow values $\{\tau_t^*\}$ and $\{\mu_t^*\}$ of Proposition III.1, to accommodate the hiring in period s.

Corollary III.3

$$M_t^* = \begin{cases} \min\{I_t, M_f\}, & t < s, \\ \min\{I_s - S_M, M_g\}, & t = s, \\ \min\{I_t, M_h\}, & t > s. \end{cases}, \quad T_t^* = \begin{cases} J, & t < s, \\ J - S_T, & t = s \\ J + y, & t > s. \end{cases};$$

$$\mu_t^* = \delta^{t-1}\left(\prod_{k=t+1}^{N} \frac{\alpha K (M_k^*)^{\alpha}(T_k^*)^{\beta}}{M_k^*}\right)\left(\frac{\alpha K (M_t^*)^{\alpha}(T_t^*)^{\beta}}{M_t^*} - 1\right) \forall t,$$

$$\tau_t^* = \delta^{t-1}\left(\prod_{k=t+1}^{N} \frac{\alpha K (M_k^*)^{\alpha}(T_k^*)^{\beta}}{M_k^*}\right)\left(\frac{\beta K (M_t^*)^{\alpha}(T_t^*)^{\beta}}{T_t^*}\right) \forall t,$$

where,

$$M_f \equiv \left(\delta\alpha K J^{\beta}\right)^{\frac{1}{1-\alpha}}, \quad M_g \equiv \left(\delta\alpha K (J - S_T)^{\beta}\right)^{\frac{1}{1-\alpha}}, \quad M_h \equiv \left(\delta\alpha K (J+y)^{\beta}\right)^{\frac{1}{1-\alpha}}.$$

Suppose hiring takes place. Then, the firm goes through three consecutive phases: (i) production without an employee, (ii) production during the hiring period, and (iii) production with the employee. Starting with an initial cash position I, the optimal cash positions at the beginning of the next period when following the prescriptions of Corollary III.3 are denoted as $f(I)$, $g(I)$, and $h(I)$, following the sequence $f \rightarrow g \rightarrow h$:

$$t < s: \quad f(I) = \begin{cases} \delta K I^{\alpha} J^{\beta} & \text{if } I < M_f \\ I + \delta(K M_f^{\alpha} J^{\beta} - M_f) & \text{otherwise,} \end{cases} \tag{III.6}$$

$$t = s: \quad g(I) = \begin{cases} \delta K (I - S_M)^{\alpha} (J - S_T)^{\beta} & \text{if } I - S_M < M_g \\ I - S_M + \delta(K M_g^{\alpha} (J - S_T)^{\beta} - M_g) & \text{otherwise,} \end{cases} \tag{III.7}$$

$$t > s: \quad h(I) = \begin{cases} \delta K I^{\alpha} (J + y)^{\beta} - w & \text{if } I < M_h \\ I + \delta(K M_h^{\alpha} (J + y)^{\beta} - M_h - w) & \text{otherwise.} \end{cases} \tag{III.8}$$

The functions f, g, h are continuously differentiable with respect to I because their derivative with respect to I at M_f, M_g, and M_h are 1. Note that for any hiring decision, the available cash I will change in each period according to the sequence $f \rightarrow g \rightarrow h$. Using these functional forms to represent revenue earned, we next examine when the entrepreneurs should hire their first employee, and the impact on the shadow values.

In order to model the resource drain of hiring, we assume that the combination of setup cost and setup time is sufficiently high such that the profit earned during the hiring period is nonpositive, formally addressed next.

Assumption III.2 $(S_T, S_M) \in \{(S_T, S_M) | S_M \geq K M_g^{\alpha} (J - S_T)^{\beta} - M_g\}$

We note that while the Assumption III.2 characterizes a sufficient condition that simplifies our analysis, it is not a necessary condition for the results of our paper.

4.2 Optimal Timing of Hiring

When is the optimal time s^* in which to hire the employee? If the entrepreneur hires too early, the resource drain caused by hiring may curb the necessary growth momentum the firm needs to maximize the objective $\delta^N I_N$. On the other hand, the entrepreneur should not hire too late when her time is valuable and when the benefit of time gained from hiring cannot be offset the time and money invested for hiring. An example illustrating the cash position over time as a result of the hiring decisions is shown in Figure III.2.

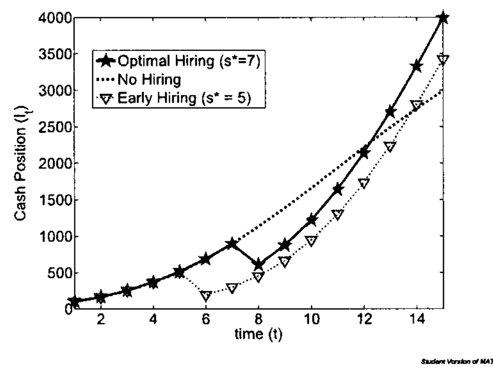

Figure III.2: Illustration of cash position dynamics as a result of hiring, with parameters $K = 1.1, J = 20, \alpha = .9, \beta = .3, \delta = .95, S_M = 400, S_T = 15, y = 10, w = 30$.

In order to formally characterize the associated tradeoffs in hiring, and to gain insights into the optimal hiring period s^* and how it is influenced by the hiring parameters (S_M, S_T, w, y), we formulate the following dynamic program representing an optimal stopping time problem. We point out that a firm with 1 employee eventually overtakes the firm with no employees as long as the period left $N - t$ is large enough, characterizing the necessity for hiring. Thus, in order to focus on the timing decision, we will assume that the remaining period is large enough so that the break-even point is always reached after hiring.

Let k denote the period-to-go, and the two dimensional variable (H,I) denote the state in each period, where $H \in \{0,1\}$ denote whether or not an employee has been hired, and I represents the available cash. For $k = 1,\ldots,N$, we thus have:

$$V_k(0,I) = \max\{\delta V_{k-1}(0,f(I)), \quad \delta V_{k-1}(1,g(I))\}, \qquad V_0(0,I) = I, \quad \text{(III.9)}$$

$$V_k(1,I) = \delta V_{k-1}(1,h(I)), \qquad V_0(1,I) = I. \text{(III.10)}$$

Because there is no further hiring decision to be made once an employee has been hired, the key decision is to determine when $V_{k-1}(0,f(I)) < V_{k-1}(1,g(I))$. For this, we use the interchange argument in the sequence of functions $f \to \cdots \to f \to g \to h \to \cdots \to h$ to determine where the function g should be in the sequence. We next present two lemmas which are necessary for applying the interchange argument.

Lemma III.2 *$V_k(1,I)$ is increasing in I, $\forall$ k.*

Lemma III.3 *Suppose Assumption III.2 holds. Then, $g(f(I)) - h(g(I))$ is strictly decreasing and continuously differentiable in I.*

Again, we note that Assumption III.2 is not a necessary condition for Lemma III.3. For example, the condition $S_T = S_M = 0$ would satisfy Lemma III.3.

We now state the following theorem, which characterizes the optimal hiring decision.

Theorem III.2 *Let I^* be the unique threshold capital for which $g(f(I^*)) = h(g(I^*))$. Then*

$$V_{k-1}(0,f(I)) < V_{k-1}(1,g(I)) \quad \textit{if and only if} \quad I > I^*.$$

The Theorem characterizes the threshold structure of the optimal policy, namely that it is optimal to hire if and only if the available cash I is above a threshold I^* (i.e. $I > I^*$). The following corollary follows from Theorem III.2 directly.

Corollary III.4

(i) Without setup cost or setup time, i.e. if $S_T = S_M = 0$, it is optimal to hire *if and only if $h(I) > f(I)$.*

(ii) If $g(f(M_f)) < h(g(M_f))$, then it is optimal to hire during the bootstrapping phase, i.e. $s^* < N^*$.

The first part of Corollary III.4 shows that concerning the hiring of employees with no significant setup cost or setup time (i.e. commodity labor), the optimal time to hire is when the revenue earned in the current period is higher with the help than without the help. That is, a one-step look ahead policy is optimal in this case. The second part establishes a sufficient condition under which the optimal period to hire is during the bootstrapping phase (i.e. when $\mu_t^* > 0$).

Hiring, which trades off time and money, need not necessarily occur when time is more valuable than money. For example, take $S_M = S_T = 0$ for simplicity and consider the tradeoff between y and w. By Corollary III.4(i), we have that I^* is when $h(I) = f(I)$, or

$$f(I^*) = h(I^*) \Leftrightarrow K(I^*)^{\alpha} J^{\beta} = K(I^*)^{\alpha}(J+y)^{\beta} - w \Leftrightarrow I^* = \left(\frac{w}{K\{(J+y)^{\beta} - J^{\beta}\}}\right)^{\frac{1}{\alpha}}.$$

Thus, as $w \longrightarrow 0$ or $y \longrightarrow \infty$, I^* can be made arbitrarily small, i.e. small enough such that

$$\frac{\alpha K J^{\beta}}{(I^*)^{1-\alpha}} - \frac{\beta K(I^*)^{\alpha}}{J^{1-\beta}} > 1, \quad \Leftrightarrow \quad \mu^* > \tau^*.$$

Hence, it may be optimal to hire even if money is more valuable than time. In other words, if additional time can be earned with minimal setup time and setup and variable costs (e.g. via process improvement), the prevailing shadow values is not relevant to the timing decision.

Under less extreme values of w and y, hiring takes place when additional time is more valuable than additional money. In general, the gap between the shadow values of time and money shadow values decreases after hiring as a result of trading off of money against time. How does hiring in the optimal period s^* influence the shadow values of money and time? The next proposition shows the implication to the difference between the shadow values of time and money after hiring.

Proposition III.2 *Let τ_t^s, μ_t^s denote the shadow prices of time and money respectively in period t if the entrepreneur hires in period s. Let s^* be the optimal hiring period. Then, for any $s \neq s^*$, $\tau_t^{s^*} \geq \tau_t^s$ and $\mu_t^{s^*} \leq \mu_t^s$ for all $t > \max\{s^*, s\}$.*

The proposition states a somewhat counter-intuitive result, namely that when hiring is made at the optimal time, the difference between the shadow values $\tau^* - \mu^*$ is *maximized* in every subsequent period. That is, given that hiring takes place at some point, the optimal timing of the hiring decision is such that the bottleneck nature of time, relative to money, is the strongest. The intuition is the following. As a result of optimal hiring, more money is accumulated during the growth phase than hiring in any other period, and the terminal value of the firm is maximized. Consequently, the shadow value of money is lower under the optimal hiring decision as there is more money available, and with access to more money, the shadow price of time becomes in contrast more valuable.

Gifford (1992) discusses the role that entrepreneur's limited attention plays in the evolving organizational structure. In particular, she argues that as the entrepreneurs free up more time by delegating current operations to managers, they can focus more attention on growth related product innovations to grow the firm. As a result however, the newly created demand for the entrepreneur's attention may be overwhelming, which makes them delegate further entrepreneurial responsibilities. While Proposition III.2 does not address whether or not the entrepreneurs should hire, given that the hiring is necessary it shows that the optimal timing of hiring is when the entrepreneur would feel the most overwhelmed by the newly created demand for the entrepreneur's time. An example comparing the shadow values for optimal timing and suboptimal timing is shown in Figure III.3.

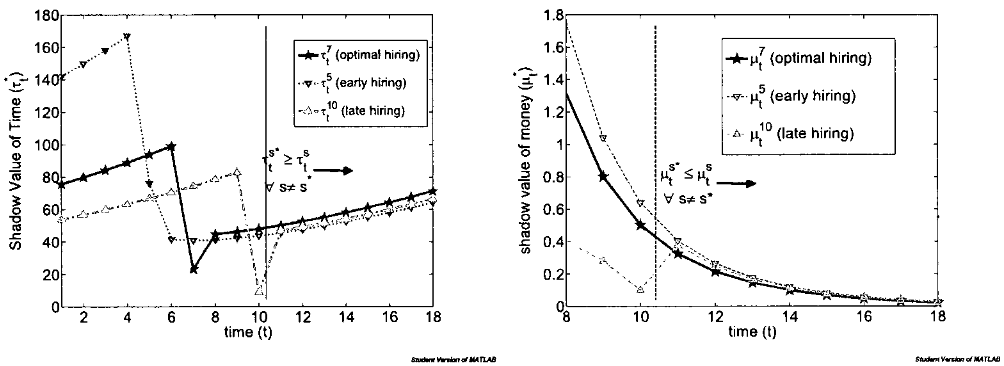

Figure III.3: The shadow values of time and money as a function of the hiring times. The optimal timing results in the largest eventual τ and the smallest μ. Parameters: $K = 1.1, I_0 = 100, J = 20, \alpha = .9, \beta = .3, \delta = .95, N = 18, S_M = 400, S_T = 10, y = 10$, $w = 30$.

4.3 Comparative Statics

In this section, we examine the comparative statics of the optimal cash threshold I^*. The next proposition examines how the hiring threshold I^* changes with respect to parameters S_M, S_T, w, y, generating insights to the optimal timing of the hiring.

Proposition III.3

$$
\begin{array}{lll}
(i) & \frac{\partial I^*}{\partial y} < 0 & \forall y \quad \textit{(expedite hiring)}, \\
(ii) & \frac{\partial I^*}{\partial w} > 0 & \forall w \quad \textit{(delay hiring)}, \\
(iii) & \frac{\partial I^*}{\partial S_M} > 0 & \forall S_M \quad \textit{(delay hiring)}, \\
(iv) & \frac{\partial I^*}{\partial S_T} > 0 & \textit{if and only if} \quad S_T > J - \chi^* \quad \textit{(delay/expedite hiring)},
\end{array}
$$

in which,

$$
\chi^* = \begin{cases}
\left(\frac{\alpha K^\alpha (I - S_M)^{\alpha^2} (J+y)^\beta}{(K I^\alpha J^\beta - S_M)^\alpha} \right)^{\frac{1}{\beta(1-\alpha)}} & \textit{if} \quad I^* < f(I^*) < M_g + S_M, \quad g(I^*) < M_h, \\
\left(\frac{\alpha^2 \beta K (I^* - S_M)^{\alpha^2} (J+y)^\beta}{(\delta\alpha)^{\frac{1}{1-\alpha}}} \right)^{\frac{1-\alpha}{\beta(1-\alpha+\alpha^2)}} & \textit{if} \quad I^* < M_g + S_M < f(I^*), \quad g(I^*) < M_h, \\
\left(\frac{\alpha\beta K (I^* - S_M)^\alpha}{(\delta\alpha K)^{\frac{1}{1-\alpha}}} \right)^{\frac{1-\alpha}{\beta\alpha}} & \textit{if} \quad I^* < M_g + S_M < f(I^*), \quad g(I^*) > M_h, \\
0 & \textit{else}.
\end{cases}
$$

The next corollary gives insight to how the hiring threshold changes with respect to J for a particular case of $S_T = S_M = 0$.

Corollary III.5 *Suppose $S_T = S_M = 0$. Then, $\frac{\partial I^*}{\partial J} > 0$ if and only if $\beta < 1$.*

The human resources literature documents the difficulty of hiring and its related resource drain to the firm (Tansky and Heneman 2006). Our model takes the hiring-associated costs as exogenous parameters and prescribes how the entrepreneur can minimize the hiring-related resource drain and lost growth momentum by controlling the timing of the hiring.

In particular, the proposition prescribes that if the potential employee demands higher wages or the upfront cost of hiring increases, hiring should be delayed to conserve the growth momentum by not diverting valuable cash away from revenue generation activities. On the other hand, if the additional time created due to the help of the employee increases, hiring should be expedited as the additional time can be used to fuel growth. However, if the upfront time associated with hiring (e.g. screening, training) is sufficiently large (small), the proposition prescribes delay (expedite) of hiring if the upfront time increases further.

The setup time $J-\chi^*$ depicts the threshold where the tradeoff between the need to preserve the growth momentum and the need to hire before the shadow value of time becomes too large intersect. If the setup time were sufficiently large ($S_T > J-\chi^*$), hiring slows the growth momentum, and thus if the setup time were to increase further, the momentum would be curbed even more. Thus, the hiring should be delayed. However, if the setup time were sufficiently small ($S_T < J-\chi^*$), the minor curbing of the growth momentum is offset by the increase in the growth rate over the remaining time horizon. Thus, if the setup time were to increase, the effect on the growth momentum can be offset by hiring earlier to reap the benefit of the accelerated growth for longer remaining time horizon.

To better understand the nonmonotonicity of the optimal hiring threshold I^* with respect to setup time S_T, we examine the iso-curves for the set of values of $\{(S_M, S_T)\}$ and $\{(y, S_T)\}$ that leads to the same hiring threshold capital I^*. The contours for the setup costs and variable costs are shown in Figure III.4. In the figure on the left hand side, we see that the iso curves are increasing in S_T initially and starts to decrease after a threshold point $J-\chi^*$. In other words, for a constant S_M, the hiring threshold I^* decreases (expedite hiring) as long as $S_T < J-\chi^*$ and starts increasing (delay hiring) for $S_T > J-\chi^*$. As S_M increases, I^* increases, consistent with Proposition III.3(iii). In the figure on the right hand side, note that as y increases, the I^* decreases (expedite hiring). However, we see that for a constant y, an increase in S_T initially decreases I^* (expedite hiring), but after $S_T \approx 6$, the I^* is increasing in S_T.

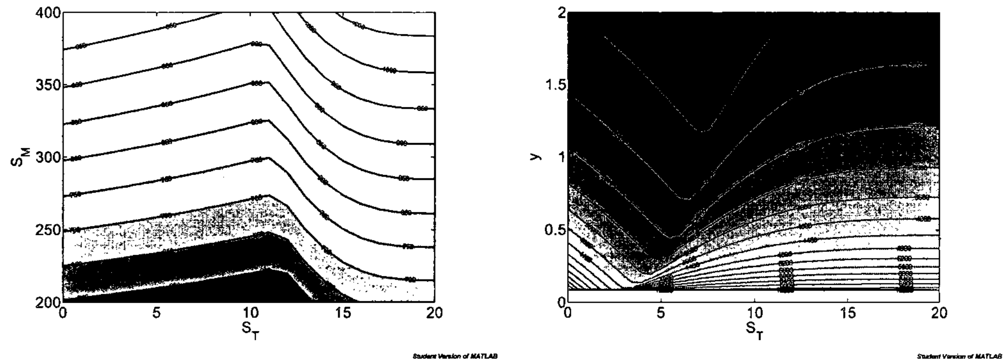

Figure III.4: Illustration of iso-curves. In the left figure ($K = 1.1, \delta = .95, \alpha = .9, \beta = .3, N = 18, y = 10, w = 30$), we illustrate the contours for setup costs. The north represents higher values of I^*, i.e. delay hiring. In the right figure, we illustrate the contours for the variable costs ($K = 1.1, \delta = 1, I = 3, J = 20, \alpha = .9, \beta = .3, N = 18, S_M = 400, w = 10$). The south represents a higher value of I^*, i.e. delay hiring.

The difference in sensitivity of the hiring threshold I^* with respect to the setup time S_T and setup cost S_M highlights the importance of differentiating the two concepts for hiring decisions in the entrepreneurial contexts. In particular, setup time is often converted into setup costs in staffing models. For example, Gans and Zhou (2002) suggest that the fixed hiring cost "typically includes advertising for, interviewing, and testing of job applicants when appropriate. It may also include one-time training costs that are independent of wages." In our context, if the setup time were treated as setup cost, then an increase in setup time would always imply that hiring should always be delayed, which is precisely contrary to the prescription of our model.

5 Extensions

In this section, we extend our model to illustrate the robustness of our results. First, we generalize the production function to the constant-elasticity-of-substitution (CES) production function, of which the Cobb-Douglas production function is a special case and demonstrate that the results of Theorem III.1 continue to hold. We then extend our model to incorporate hiring of multiple employees, where the order of the given employees are predetermined, and identify the moderate conditions under which the results of Theorem III.2 as well as the comparative statics of the hiring parameters continue to hold.

5.1 Generalization to CES Production Function

The constant-elasticity-of-substitution (CES) production function (Arrow et al. 1961) and the resulting profit function is given by the following respective expressions:

$$R(M,T) = K\left(pT^q + (1-p)M^q\right)^{\frac{r}{q}}, \qquad \Pi(M,T) = K\left(pT^q + (1-p)M^q\right)^{\frac{r}{q}} - M,$$

where $p = \frac{\alpha}{\alpha+\beta}$, and $r = \alpha + \beta$.

Following the definitions of Arrow et al. (1961), we will refer to K as the efficiency parameter, p as the distribution parameter, and q as the substitution parameter. When $r = 1$, the revenue function converges to the Leontieff function (perfect complement), i.e. $K\min\{M,T\}$ when $q \longrightarrow -\infty$; to the Cobb-Douglas function (complement) $KM^{\alpha}T^{\beta}$ when $q \longrightarrow 0$; and to a linear function (perfect substitute) when $q = 1$. It is easily verified that the revenue $R(q)$ (and thus $\Pi(q)$) is increasing in $q \in (-\infty, 1]$ (as money and time becomes more substitutable). The following Corollary generalizes the optimal shadow values $\{\mu_t^*\}$ and $\{\tau_t^*\}$.

Corollary III.6 *The expressions for μ_t^* and τ_t^* can be generalized to CES functions as follows:*

$$\mu_t^* = \left(\prod_{k=t+1}^{N-1} \frac{\partial R(M_k^*, T_k^*)}{\partial M_k^*}\right) \left(\frac{\partial \Pi(M_t^*, T_t^*)}{\partial M_t^*}\right),$$
$$\tau_t^* = \left(\prod_{k=t+1}^{N-1} \frac{\partial R(M_k^*, T_k^*)}{\partial M_k^*}\right) \left(\frac{\partial \Pi(M_t^*, T_t^*)}{\partial T_t^*}\right).$$

The following lemma establishes the sufficient conditions for the concavity of $R(M,T)$ with respect to M.

Lemma III.4 *$R(M,T)$ is concave in M if*

(i) $\alpha+\beta \leq 1, \quad \forall q,$

(ii) $\alpha+\beta > 1, \quad q=0$ *and* $\alpha < 1$ *(Cobb-Douglas),*

(iii) $\alpha+\beta > 1, \quad q<0$ *and* $I_0 \geq \left\{\left(\frac{\beta}{\beta-(1-\alpha)}\right)\left(\frac{1-q}{\alpha}\right)\right\}^{1/q},$

(iv) $\alpha+\beta > 1, \quad q>0$ *and* $I_N \leq \left\{\left(\frac{\beta}{\beta-(1-\alpha)}\right)\left(\frac{1-q}{\alpha}\right)\right\}^{1/q}.$

We examine the evolution of shadow values $\mu_t^*(q)$ and $\tau_t^*(q)$ over time as the substitution parameter $q \in (-\infty, 1]$ increases and show the robustness of Theorem III.1.

Theorem III.3 *As long as $R(M,T)$ remains concave in M,*

$$\frac{\mu_{t+1}^*}{\mu_t^*} \leq \frac{\tau_{t+1}^*}{\tau_t^*}.$$

Thus, the shadow value of time τ_t^* becomes greater than the shadow value of money μ_t^*.

In particular, the shadow value of money remains decreasing as described in Theorem III.1 but the shadow value of time may not necessarily increase. In fact as the substitutability between time and money q increases, the dynamic evolution of the shadow value of time resembles more of that of the shadow value of money. For example, when $q = 1$, the shadow value of time decreases like that of money. This is because the difference between time and money disappears and time behaves as if it gets stored like money because it can be saved into the next period by being converted into money. In other words, time can substitute for money, but money cannot substitute for time.

Nevertheless, when the shadow value of time decreases, it does so slower than the shadow value of money, thus ultimately becoming greater than that of money. Thus, the shift in bottleneck from money to time characterized by Theorem III.1 is preserved for when the returns to scale are decreasing (Lemma III.4(i)), or under specific conditions on the initial and final wealth (Lemma III.4(iii),(iv)).

5.2 Multiple Hiring

As the entrepreneurial firm continues to grow, the gap between the shadow values between time and money increases, making it necessary to hire additional employees. In this section, we examine the robustness of our results to multiple hiring and provide insights to the entrepreneurs' multiple hiring decisions when the sequence of employees are predetermined. To reflect the resource constraints of the entrepreneurial firms, we restrict the hiring in each period to one employee.

Let y_ℓ and w_ℓ denote the time gained and the wage expended by hiring the ℓ^{th} employee. Thus, after hiring the ℓ^{th} employee, the available time increases from $J +$

$\sum_{j=1}^{\ell-1} y_j$ to $J+\sum_{j=1}^{\ell} y_j$. Furthermore, let S_{M^ℓ} and S_{T^ℓ} denote the hiring setup cost and setup time respectively of the ℓ^{th} employee.

Following the notations of §4.1, we will denote the optimal cash position in the ensuing period as a result of optimal time and money investments with ℓ employees as $h_\ell(I)$ (with $h_0 \equiv f$); and that during the hiring period in which you hire the ℓ^{th} employee by $g_\ell(I)$, which are respectively shown next.

$$h_\ell(I) = \begin{cases} KI^\alpha(J+\sum_{j=1}^{\ell} y_j)^\beta - \sum_j^\ell w_j & \text{if } I < M_{h_\ell} \\ I + (KM_{h_\ell}^\alpha (J+\sum_{j=1}^{\ell} y_j)^\beta - M_{h_\ell} - \sum_{j=1}^{\ell} w_j) & \text{otherwise,} \end{cases}$$

$$g_\ell(I) = \begin{cases} K(I-S_M)^\alpha(J+\sum_{j=1}^{\ell-1} y_j - S_T^\ell)^\beta & \text{if } I - S_M^\ell < M_{g_\ell} \\ I - S_M^\ell + (KM_{g_\ell}^\alpha (J+\sum_{j=1}^{\ell-1} y_j - S_T^\ell)^\beta - M_{g_\ell}) & \text{otherwise,} \end{cases}$$

$$\text{where} \quad M_{h_\ell} \equiv \left(\delta\alpha K(J+\sum_{j=1}^{\ell} y_j)^\beta\right)^{\frac{1}{1-\alpha}}, \quad M_{g_\ell} \equiv \left(\delta\alpha K(J+\sum_{j=1}^{\ell-1} y_j - S_T^\ell)^\beta\right)^{\frac{1}{1-\alpha}}.$$

Moreover, we assume that each setup cost and setup time for hiring the ℓ^{th} employee satisfy the Assumption III.2, formally stated next:

Assumption III.3

$(S_{T^\ell}, S_{M^\ell}) \in \{(S_{T^\ell}, S_{M^\ell}) | S_{M^\ell} \geq KM_{g_\ell}^\alpha (J - S_{T^\ell})^\beta - M_{g_\ell}\}$, $\forall \ell \in \{0,1,\ldots,\bar{\ell}\}$.

The DP formulation of (III.9)-(III.10) can be extended to multiple hiring by expanding the state $H \in \{0,1\}$ to $H \in \{0,1,2,\ldots,\bar{\ell}\}$, where $\bar{\ell}$ denote the maximum number of employees to hire. Again, we point out that a firm with $(\ell+1)$ employees eventually overtakes the firm with ℓ employees as long as the remaining periods $N-t$ is long enough. In order to focus on the timing decision, as done in §4.2, we will assume that the remaining period is sufficiently large so that the break-even point is reached after hiring the $(\ell+1)^{th}$ employee. We have:

$$V_k(\ell,I) = \max\{\delta V_{k-1}(\ell,h_\ell(I)), \quad \delta V_{k-1}(\ell+1,g_{\ell+1}(I))\}, \quad V_0(\ell,I)=I, \forall\ell<\overline{\ell},$$
$$V_k(\overline{\ell},I) = \delta V_{k-1}(\overline{\ell},h_{\overline{\ell}}(I)), \qquad V_0(\overline{\ell},I)=I.$$

Analogous to §4.2, for each ℓ, $V_k(\ell,I)$ is increasing in I for all k (Lemma III.2); and the expression $g_\ell(h_\ell(I)) - h_{\ell+1}(g_\ell(I))$ is strictly decreasing and continuously differentiable in I if Assumption III.3 holds (Lemma III.3). Thus, the $\overline{\ell}$ independent threshold capital levels $\{I_\ell^*\}$ such that $g_\ell(h_\ell(I_\ell^*)) = h_{\ell+1}(g_\ell(I_\ell^*))$, i.e. for a firm which currently has available time $J+\sum_{j=1}^{\ell} y_j$ (ℓ employees) and decides to hire exactly one more employee (the $(\ell+1)^{th}$ employee), are well defined.

The thresholds I_ℓ^*'s need not be necessarily increasing in ℓ. For example, if $S_M = S_T = 0$, then by Corollary III.5, we have that $I_{\ell+1}^* < I_\ell^*$ $\forall\ell$ whenever $\beta > 1$ because the available time with $\ell+1$ employees is greater than that with ℓ employees. If $\exists\ell, I_{\ell+1}^* < I_\ell^*$, it may be optimal to hire the ℓ^{th} employee before the period it would have been optimal to hire her if she were the last employee (i.e. hire when $I < I_\ell^*$), so as to avoid further delaying the hiring of the $(\ell+1)^{th}$ employee.

In practice however, the hiring sequence is such that the setup costs $\{S_{M^\ell}\}$ and the variable cost $\{w_\ell\}$ is increasing in ℓ. This is because the task that entrepreneurs need to delegate evolves from low-skilled to sophisticated. For example, entrepreneurs typically first hire undergraduates to delegate tasks such as programming or internet research, then MBA's to delegate tasks related to marketing or accounting, before searching for senior level VPs to delegate tasks related to strategic growth. By Proposition III.3, if the sequences $\{S_{M^\ell}\}$ and $\{w_\ell\}$ were sufficiently increasing, the sequence of thresholds $\{I_\ell^*\}$ would be increasing. Thus, we assume the following:

Assumption III.4 *The sequence of employee parameters* $\{(S_{M^\ell}, S_{T^\ell}, w_\ell, y_\ell)\}$ *is such* *that* $I_1^* < I_2^* < \cdots < I_\ell^*$.

We have the following theorem.

Theorem III.4 *Suppose Assumptions III.3 & III.4 hold. Then, it is optimal to hire the* ℓ^{th} *employee if and only if* $I > I_\ell^*$. *In other words,*

$$V_k(\ell,I) = \begin{cases} \delta V_{k-1}(\ell,h_\ell(I)) & \text{if } \; I < I_\ell^* \\ \delta V_{k-1}(\ell+1,g_\ell(I)) & \text{otherwise} \end{cases}$$

The theorem states that if the employee hiring is sequenced such that the independent thresholds $\{I_\ell^*\}$ are increasing, it is optimal for the entrepreneur to employ the threshold policy for hiring each employee. In other words, it is optimal to hire the ℓ^{th} employee without any regards to future hiring decisions. In such case, because each hiring decision depends only on the thresholds I_ℓ^*'s, the comparative statics derived in Proposition III.3 hold for each hiring decision.

Examples of optimal hiring for the two cases (when Assumption III.4 does not hold and when it does) are illustrated in Figure III.5. There are two employees A $((S_M,S_T,y,w) = (200,15,15,10))$ and B $((S_M,S_T,y,w) = (400,19,5,20))$. On the left hand side, the predetermined sequence of hiring is $B \rightarrow A$, which gives the independent hiring thresholds of $I_1^* = 1442$ and $I_2^* = 538$. In such case, we see that the first employee is hired before the cash position is above the independent threshold I_1^* (in period 6), and the second employee is not hired until period 7 even though the threshold level I_2^* has been surpassed in period 3. On the right hand side, the predetermined sequence of hiring is $A \rightarrow B$, which gives increasing independent hiring thresholds of $I_1 = 456$ and $I_2 = 1953$. Thus, we observe that each employee is hired as soon as the cash position exceeds the respective threshold levels (in period 3 and in period 8).

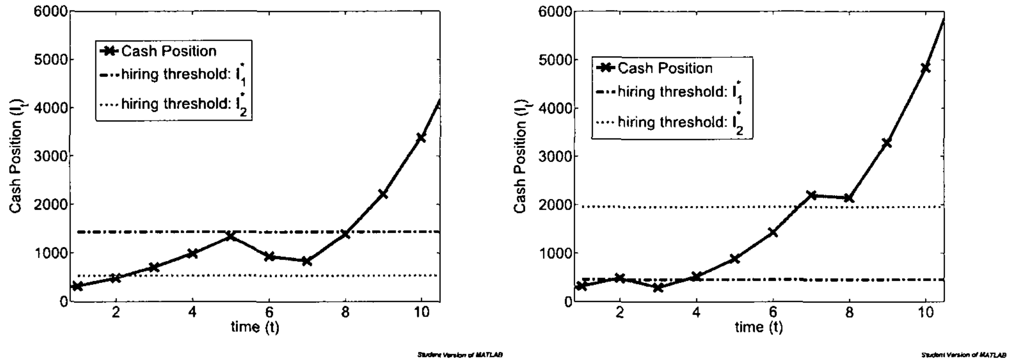

Figure III.5: Evolution of cash position under optimal timing of hiring under two different sequences. Parameters of the firm: $K = 1.1, I = 300, J = 20, \alpha = .9, \beta = .3, \delta = .99, N = 11$. Two employees: A ($S_{M^A} = 200, S_{T^A} = 15, y_A = 15, w_A = 10$) and B ($S_{M^B} = 400, S_{T^A} = 19, y_B = 5, w_A = 20$). The left hand side hires $B \rightarrow A$ ($I_1^* = 1442$ and $I_2^* = 538$); the right hand side hires $A \rightarrow B$ ($I_1^* = 456$ and $I_2^* = 1953$).

6 Concluding Remarks

In this paper, we present an entrepreneurial firm's production function and examine how the complementary inputs of time and money interact as the firm grows. We now summarize our major findings.

First, during the bootstrapping phase, the shadow value of time is increasing whereas the shadow value of money is decreasing, ultimately making time the key bottleneck resource. This is driven by the physical difference between time and money as a resource, and the structure of the production function. In contrast to money, which can be reinvested into the firm, time is physically limited each period. Thus, the available money increases but the available time remains constant during growth, making time relatively more scarce than money in subsequent periods. Secondly, because both time and money are required to generate revenue (complementary resources), the relative value of time is greater when there is more available money to invest. Thus, the value of additional time increases as the supply of money increases. We show that this shift in the bottleneck from money to time is robust to the functional form of the production function.

Second, we establish that there is a unique cash level threshold above which it is optimal to hire. We also establish that under certain sequence of hiring, this thresholds-based policy remains optimal for multiple hiring decisions. This hiring threshold is non-monotonic in the hiring setup time due to the tradeoff between the need to preserve the growth momentum and the need to hire before the shadow value of time becomes large. In particular, if the setup time is above a certain threshold, hiring curbs the growth momentum and thus if the setup time were to increase further, the model prescribes delaying the hiring. On the other hand, if the setup time is below the threshold, the minor curbing of the growth momentum is offset by the increased growth over the remaining time horizon. Thus, if the setup time is small but were to increase, the model prescribes expediting hiring to accelerate growth and reap the benefits further.

Third, we find that when the entrepreneur hires at the optimal time, the gap between the shadow values of time and money is ultimately maximized. This is because, as a result of optimal hiring, more money is accumulated during the growth phase than hiring in any other period. Consequently, the shadow value of money is the lowest because there is more money available, and with access to more money, the shadow price of time becomes in contrast more valuable.

Finally, our model points out the importance of differentiating setup time and setup cost when making hiring decisions. Although staffing models for large firms often aggregate the setup time into setup cost because dedicated human resources department exist and the time can be easily converted into their hourly wages, there exists a fundamental difference between hiring setup time and cost in an entrepreneurial context.

This paper does ignore some important factors relevant to hiring in the entrepreneurial context. For instance, the employees and their parameters are assumed to be exogenous, whereas they could be endogenously determined by the amount of setup time or setup cost the entrepreneur invests. Treating the investment of setup time and setup costs as decision variables in this regards can be an interesting extension. Moreover the time gained by hiring an employee is assumed to be known and independent of the existence of other employees. It would be worthwhile to incorporate uncertainty into the employee's contribution or model the turnover rate of employees and examine how it affects the entrepreneur's hiring decision. Furthermore, it would be valuable to gain insights into the optimal hiring sequence of the employees, and how it is influenced by the fit between employees. Generalization of some of our results to address such issues, although challenging, would be worthwhile.

Appendix

Proof of Proposition III.1 We first construct the optimal primal solution (M_t^*, T_t^*). Because the constraints on the money (III.3) is increasing in the profit Π_t, it is optimal to invest (M_t, T_t) that will maximize the single period profit $\Pi_t(M_t, T_t)$. Since

$$\begin{aligned}
\frac{\partial}{\partial T_t}\Pi_t(M_t,T_t) &= \frac{\partial}{\partial T_t}\left\{\delta K M_t^{\alpha} T_t^{\beta} - M_t\right\} = \frac{\beta\delta K M_t^{\alpha}}{T_t^{1-\beta}} > 0, \quad \forall T_t, \quad \text{and} \\
\frac{\partial}{\partial M_t}\Pi(M_t,J) &= \frac{\partial}{\partial M_t}\{\delta K M_t^{\alpha} J^{\beta} - M_t\} = \frac{\alpha\delta K T_t^{\beta}}{M_t^{1-\alpha}} > 0 \quad \Leftrightarrow \quad M_t < M_f,
\end{aligned}$$

we have $M^* = \min\{I_t, M_f\}$, and $T^* = J$, where $M_f \equiv \left(\delta\alpha K J^{\beta}\right)^{\frac{1}{1-\alpha}}$.

The constraint gradient at the vector $\{(M_t^*, T_t^*)\}$ is linearly independent – and hence *regular* – because for each constraint (III.3) and (III.4), the new variable M_t and T_t are introduced. Therefore, a unique Lagrange multiplier vector exists (Proposition 3.3.1, Bertsekas 1999), where the gradient of the Lagrangian is zero. Moreover, at this point, the complementary slackness holds because $T_t^* = J\ \forall t$, $\tau_t > 0\ \forall t$, and $\mu_t > 0$ if $M_t = I_t < M_f$ and $\mu_t = 0$ if $M_t = M_f \geq I_t$.

We find the expressions for the Lagrange multipliers $\{\mu_t^*, \tau_t^*\}$. For simplicity, we will ignore the superscript *. From (III.2)–(III.5), we have the following Lagrangian:

$$\begin{aligned}
L &= \sum_{t=0}^{N-1} \delta^t \left\{\delta K M_t^{\alpha} T_t^{\beta} - M_t\right\} \\
&\quad + \sum_{t=0}^{N-1} \mu_t \left(\left(\frac{1}{\delta}\right)^t I_0 + \sum_{k=0}^{t-1} \left(\frac{1}{\delta}\right)^{t-k} \left\{\delta K M_k^{\alpha} T_k^{\beta} - M_k\right\} - M_t \right) + \sum_{t=0}^{N-1} \tau_t (J - T_t).
\end{aligned}$$

We will proceed in three steps: (i) derive the expression for τ_t, (ii) derive the expression for μ_t, then (iii) derive the expression for the coefficient that appears in both τ_t and μ_t.

i) First, taking the derivative of L with respect to T_t, we have,

$$\begin{aligned}\frac{\partial L}{\partial T_t} &= \delta^{t-1}\frac{\beta}{T_t}KM_t^{\alpha}T_t^{\beta} + \left(\sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\beta}{T_t}KM_t^{\alpha}T_t^{\beta}\right) - \tau_t = 0 \\ \Rightarrow \quad \tau_t &= \left(\delta^{t-1} + \sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\beta}{T_t}KM_t^{\alpha}T_t^{\beta}\right). \end{aligned} \tag{III.11}$$

ii) Next, taking the derivative of L with respect to M_t we have,

$$\begin{aligned}\frac{\partial L}{\partial M_t} &= \delta^{t-1}\left(\frac{\alpha KM_t^{\alpha}T_t^{\beta}}{M_t} - 1\right) - \mu_t + \left(\sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\alpha KM_t^{\alpha}T_t^{\beta}}{M_t} - 1\right) = 0 \\ \Rightarrow \quad \mu_t &= \left(\delta^{t-1} + \sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\alpha KM_t^{\alpha}T_t^{\beta}}{M_t} - 1\right). \end{aligned} \tag{III.12}$$

iii) Finally, we show the following equality:

$$\left(\delta^{t-1} + \sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right) = \delta^{t-1}\prod_{k=t+1}^{N}\left(\frac{\alpha K(M_k^*)^{\alpha}(T_k^*)^{\beta}}{M_k^*}\right).$$

We do so by induction. This true for $t = N$ because

$$\left(\delta^{t-1} + \sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)_{t=N} = \delta^{N-1}. \tag{III.13}$$

Now suppose,

$$\left(\delta^{t} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-(t+1)}\mu_k\right) = \delta^{t}\prod_{k=t+2}^{N}\left(\frac{\alpha KM_k^{\alpha}T_k^{\beta}}{M_k}\right).$$

Then, we have

$$\delta^{t-1} + \sum_{k=t+1}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k$$

$$
\begin{aligned}
&= \delta^{t-1} + \left(\frac{1}{\delta}\right)\mu_{t+1} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k \\
&= \delta^{t-1} + \left(\delta^{t-1} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}} - 1\right) + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k \\
&= \delta^{t-1} + \left(\delta^{t-1} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}}\right) \\
&\qquad - \left(\delta^{t-1} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right) + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k \\
&= \left(\delta^{t-1} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-t}\mu_k\right)\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}}\right) \\
&= \left(\frac{1}{\delta}\right)\left(\delta^{t} + \sum_{k=t+2}^{N}\left(\frac{1}{\delta}\right)^{k-(t+1)}\mu_k\right)\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}}\right) \\
&= \left(\frac{1}{\delta}\right)\delta^{t}\prod_{k=t+2}^{N}\left(\frac{\alpha K M_k^{\alpha} T_k^{\beta}}{M_k}\right)\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}}\right) \\
&= \delta^{t-1}\prod_{k=t+1}^{N}\left(\frac{\alpha K M_k^{\alpha} T_k^{\beta}}{M_k}\right),
\end{aligned}
$$

where the second equality is due to equation (III.12) and second to last equality is due to equation (III.13). ■

Proof of Lemma III.1 Before $I_t < M_f \equiv \left(\delta\alpha K J^{\beta}\right)^{\frac{1}{1-\alpha}}$, the optimal decision is to put all the earned revenue back into the business. First, we have that

$$
I_1 = \left(\frac{K}{1+r} I_0^{\alpha} J^{\beta}\right), \quad \text{and} \quad I_t = \left(\frac{K}{1+r} I_{t-1}^{\alpha} J^{\beta}\right), \forall t = 0, \ldots, N.
$$

Elaborating, we have,

$$
\left(\left(\frac{K}{1+r}\right) J^{\beta}\right)^{1+\alpha+\cdots+\alpha^{N-1}} I^{\alpha^N} < M_f
$$

$$\Leftrightarrow \left(\left(\frac{K}{1+r}\right)J^{\beta}\right)^{\frac{1-\alpha^N}{1-\alpha}} I^{\alpha^N} < \left(\alpha\left(\frac{K}{1+r}\right)J^{\beta}\right)^{\frac{1}{1-\alpha}}$$

$$\Leftrightarrow (1-\alpha)\alpha^N \log I + (1-\alpha^N)\log\left(\left(\frac{K}{1+r}\right)J^{\beta}\right) < \log\left(\alpha\left(\frac{K}{1+r}\right)J^{\beta}\right)$$

$$\Leftrightarrow (1-\alpha)\alpha^N \log I - \alpha^N \log\left(\left(\frac{K}{1+r}\right)J^{\beta}\right) < \log\left(\frac{\alpha\left(\frac{K}{1+r}\right)J^{\beta}}{\left(\frac{K}{1+r}\right)J^{\beta}}\right)$$

$$\Leftrightarrow \alpha^N \log I^{1-\alpha} - \alpha^N \log\left(\left(\frac{K}{1+r}\right)J^{\beta}\right) < \log\alpha$$

$$\Leftrightarrow \alpha^N \log\left(\frac{I}{\left(\frac{K}{1+r}\right)I^{\alpha}J^{\beta}}\right) < \log\alpha$$

$$\Leftrightarrow N < \log\left(\frac{\log\left(\frac{1}{\alpha}\right)}{\log\left(\frac{\left(\frac{K}{1+r}\right)I^{\alpha}J^{\beta}}{I}\right)}\right)\frac{1}{\log\alpha}.$$

■

Proof of Corollary III.1 Because the objective function (III.2) and the constraints (III.3), (III.4) are continuously differentiable in $\{(M_t, T_t)\}$, the Hessian of Lagrangian (restricted to the vector space orthogonal to the constraint gradients at $\{(M_t^*, T_t^*)\}$) evaluated at $\{(M_t^*, T_t^*, \mu_t^*, \tau_t^*)\}$ is positive semidefinite (Proposition 3.3.1, Bertsekas 1999). Furthermore, since this Hessian has linearly independent column vectors spanning this restricted vector space (i.e. full rank), all the associated eigenvalues are strictly positive and hence the Hessian is strictly positive definite. Moreover, since strict complementary slackness holds (see Proof of Proposition III.1), the sensitivity result follows (Proposition 3.3.3, Bertsekas 1999). ■

Proof of Theorem III.1 For simplicity of notations, we omit the superscript *.

i) We have,

$$\frac{\mu_{t+1}}{\mu_t} = \delta \cdot \frac{1}{\left(\alpha K M_{t+1}^{\alpha-1} T_{t+1}^{\beta}\right)} \cdot \frac{\left(\alpha K M_{t+1}^{\alpha-1} T_{t+1}^{\beta} - 1\right)}{\left(\alpha K M_t^{\alpha-1} T_t^{\beta} - 1\right)} < \delta < 1, \quad \forall t.$$

This inequality holds since,

$$\frac{\partial \Pi_t}{\partial M_t} = \alpha K M_t^{\alpha-1} T_t^{\beta} - 1 \geq 0 \quad \Leftrightarrow \quad \alpha K M_t^{\alpha-1} T_t^{\beta} \geq 1,$$

where strict inequality holds if $I_t < M_f \equiv \left(\delta\alpha K J^{\beta}\right)^{\frac{1}{1-\alpha}}$ since $M_t^* = I_t$ and equality holds if $I_t > M_f$ since $M_t^* = M_f$. Furthermore, since Π_t is strictly concave increasing for $M_t < M_f$, $\Pi'(M_1) > \Pi'(M_2) > 0$ for $M_1 < M_2$ and $M_{t+1} > M_t$. If $I_t > M_f$, $\Pi'(M) = 0$ and the inequality becomes equality.

ii) For the ratios of τ's, we have

$$\frac{\tau_{t+1}}{\tau_t} = \frac{\delta\left(\frac{\beta K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{T_{t+1}}\right)}{\left(\frac{\alpha K M_{t+1}^{\alpha} T_{t+1}^{\beta}}{M_{t+1}}\right)\left(\frac{\beta K M_t^{\alpha} T_t^{\beta}}{T_t}\right)} = \frac{1}{\alpha}\frac{\delta M_{t+1}}{K M_t^{\alpha} J^{\beta}},$$

given that $T_{t+1} = J$. For all $t < N^* - 1$ such that $I_{t+1} < M_f$, $\frac{\delta M_{t+1}}{K M_t^{\alpha} J^{\beta}} = 1$, and we have $\frac{\tau_{t+1}}{\tau_t} = \frac{1}{\alpha} > 1\ \forall t$, i.e. the sequence $\{\tau_t\}$ is exponentially increasing in t.

For $t > N^*$ such that $M_f < I_t$, we have $\frac{\delta M_{t+1}}{K M_t^{\alpha} J^{\beta}} = \frac{\delta M_f}{K M_f^{\alpha} J^{\beta}} = \delta\alpha$. Thus, we have $\frac{\tau_{t+1}}{\tau_t} = \frac{1}{\alpha}(\delta\alpha) = \delta < 1$.

iii) For $t < N^* - 1$, since $\{\tau_t\}$ is increasing in t, it is clear that $\tau_t - \mu_t$ is increasing. Moreover, for all $t \geq N^*$, $\mu_t = 0$ and $\tau_t > 0$, and thus $\tau_t > \mu_t$. ■

Proof of Corollary III.2 Note that

$$\mu_t > \tau_t \quad \Leftrightarrow \quad \frac{\partial \Pi}{\partial M} > \frac{\partial \Pi}{\partial T}.$$

Elaborating the expression, we have

$$\left.\frac{\partial \Pi}{\partial M}\right|_{I_1,J} = \frac{\alpha}{I_1} K I_1^{\alpha} J^{\beta} - 1 > \left.\frac{\partial \Pi}{\partial T}\right|_{I_1,J} = \frac{\beta}{J} K I_1^{\alpha} J^{\beta} \quad \Leftrightarrow \quad K I_1^{\alpha} J^{\beta}\left(\frac{\alpha}{I_1} - \frac{\beta}{J}\right) > 1.$$

We have our expression after simple algebra. Since by Theorem III.1, there is a unique crossing point between the sequences $\{\tau_t^*\}$ and $\{\mu_t^*\}$, if this condition holds, i.e. $\tau_1^* > \mu_1^*$, then $\tau_t^* > \mu_t^*\ \forall t$. ■

Proof of Corollary III.3 Same structure as the proof of Proposition III.1 with changes to constraints (III.3) and (III.4) during and after the hiring period. ■

Proof of Lemma III.2. We prove by induction. It is clear that at $t = 0$, $V_0(1,I) = I$ is increasing in I. Now suppose that $V_{t-1}(1,I)$ is increasing in I, i.e., $V_{t-1}(1,I_1) > V_{t-1}(1,I_2)$, $\forall I_1 > I_2$. This is equivalent to $V_t(1,h^{-1}(I_1)) > V_t(1,h^{-1}(I_2))$. Because $h(I)$ is strictly increasing in I and positive, $h^{-1}(I)$ is also strictly increasing in I and spans all positive real numbers. ■

Proof of Lemma III.3 We have

$$g(I) - \min\{M_g + S_M, I\} \leq \Pi_g = \max_{M \leq I - S_M} \{KM^{\alpha}(J - S_T)^{\beta} - M - S_M\} \leq 0$$

where the first inequality is by definition and the second inequality is due to Assumption III.2. Thus, $g(I) \leq \min\{M_g + S_M, I\} \leq I$. Taking the derivative of $g(f(I)) - h(g(I))$ with respect to I, we have

$$\begin{aligned}
(g(f(I)) - h(g(I)))' &= g'(f(I)) \cdot f'(I) - h'(g(I)) \cdot g'(I) \\
&< g'(f(I)) \cdot h'(I) - h'(g(I))g'(I) \\
&\leq g'(I)h'(I) - h'(g(I))g'(I) \\
&= g'(I)(h'(I) - h'(g(I)) \\
&\leq 0,
\end{aligned}$$

where the first strict inequality is because $f' \leq h'$ because $y > 0$, the second inequality is because $f(I) \geq I$ because of Assumption III.1, and the final inequality is because $g(I) \leq I$. ■

Proof of Theorem III.2. By Lemma III.2, $V_t(1, g(f(I))) > V_t(1, h(g(I)))$, if and only if $g(f(I)) > h(g(I))$. Because all the previous periods prior to and subsequent to hiring have identical functional forms, we use the interchange argument (§4.5, Bertsekas

2000). Since $g(f(I)) > h(g(I)) = 0$ for $I = S_M$, $g(f(I)) - h(g(I)) = \Pi_f - \Pi_h < 0$ for large I, and because $g(f(I)) - h(g(I))$ is decreasing in I by Lemma III.3, there exists a unique I^* for which it is optimal to hire if $g(f(I)) \leq h(g(I))$. ∎

Proof of Corollary III.4.

(i) By Theorem III.2, $g(I) = f(I)$ implies that it is optimal to hire when $f(f(I)) < h(f(I))$ or $f(I) < h(I)$.

(ii) Since $g(f(I)) - h(g(I))$ is decreasing in I, by definition of I^*,

$g(f(M_f)) - h(g(M_f)) < 0$ if and only if $I^* < M_f$.

∎

Proof of Proposition III.2 We let τ_t^s and μ_t^s denote the shadow values of time and money respectively in period t if you hire in period s, and let s^* denote the optimal hiring period. We will omit $*$ for notational simplicity.

First, we show that for all $s \neq s^*$, $\mu_t^{s^*} \leq \mu_t^s$ $\forall t > \max\{s, s^*\}$. Note that $\forall k > \max\{s^*, s\}$, the available cash I_k (and hence the money investment M_k) when hired in the optimal period s^* is no smaller than when hired in period $s \neq s^*$. Because $R(M,J)$ is concave increasing in M with $\frac{\partial R(M_k,T_k)}{\partial M_k} = \frac{\alpha K M_k^\alpha T_k^\beta}{M_k} \geq 1$, the expression

$$\mu_t^* = \prod_{k=t+1}^{N} \left(\frac{\alpha K (M_k^*)^\alpha (T_k^*)^\beta}{M_k^*} \right) \left(\frac{\alpha K (M_t^*)^\alpha (T_t^*)^\beta}{M_t^*} - 1 \right)$$

is minimized $\forall t > \max\{s, s^*\}$ when hired in s^*.

Next we show that for all $s \neq s^*$, $\tau_t^{s^*} \geq \tau_t^s$ $\forall t > \max\{s^*, s\}$. Note that $\forall t > \max\{s, s^*\}$, $T_t^* = (J + y)$. Moreover, let η^s denote the largest period k in which $M_k < M_h \equiv (\delta\alpha(J+y))^{\frac{1}{1-\alpha}}$, i.e. the final bootstrapping period, when hired in period s. Because M_k's are the greatest $\forall k > \max\{s^*, s\}$ when hired in the optimal period s^*, $\eta^{s^*} \leq \eta^s$. We will consider the two exhaustive cases separately: (i) when $\eta^{s^*} = \eta^s$, and (ii) when $\eta^{s^*} < \eta^s$.

(i) $\forall t > \max\{s, s^*\}$, we have

$$\begin{aligned}
\tau_t^{s^*} &= \left(\frac{\beta K (M_t)^\alpha (T_t)^\beta}{T_t}\right) \prod_{k=t+1}^{\eta^{s^*}} \left(\frac{\alpha K M_k^\alpha T_k^\beta}{M_k}\right) \prod_{k=\eta^{s^*}+1}^{N} 1 \\
&= \left(\frac{\beta K M_t^\alpha (J+y)^\beta}{J+y}\right) \left(\frac{\alpha K M_{t+1}^\alpha (J+y)^\beta}{M_{t+1}}\right) \cdots \left(\frac{\alpha K M_{\eta^{s^*}}^\alpha (J+y)^\beta}{M_{\eta^{s^*}}}\right) \\
&= \left(\frac{\beta K M_t^\alpha (J+y)^\beta}{J+y}\right) \alpha^{\eta^{s^*}-t} \left(\frac{K M_{\eta^{s^*}}^\alpha (J+y)^\beta}{M_{t+1}}\right) \\
&= \frac{\beta \alpha^{\eta^{s^*}-t} K M_{\eta^{s^*}}^\alpha (J+y)^\beta}{J+y},
\end{aligned}$$

where the cancelations in the third equality occurs because it is optimal to bootstrap, i.e. $M_{t+2} = K M_{t+1}^\alpha (J+y)^\beta$. If $\eta^{s^*} = \eta^s$, then $M_{\eta^{s^*}}$ is greatest when hired in the optimal period, and hence the result holds.

(ii) Now let $\eta^{s^*} < \eta^s$. Then, we have

$$\tau_t^{s^*} = \frac{\beta \alpha^{\eta^{s^*}-t} K M_{\eta^{s^*}}^\alpha (J+y)^\beta}{J+y}, \quad \text{and} \quad \frac{\beta \alpha^{\eta^s - t} K M_{\eta^s}^\alpha (J+y)^\beta}{J+y}.$$

Thus, we will determine whether or not the following holds:

$$\frac{\tau_t^{s^*}}{\tau_t^s} = \alpha^{\eta^{s^*}-\eta^s} \left(\frac{M_{\eta^{s^*}}}{M_{\eta^s}}\right)^\alpha > 1 \quad \Leftrightarrow \quad \left(\frac{M_{\eta^{s^*}}}{M_{\eta^s}}\right)^\alpha > \alpha^{\eta^s - \eta^{s^*}}.$$

We will use the following two properties: (a) $M_h > \max\{M_{\eta^{s^*}}, M_{\eta^s}\}$ and (b) $\min\{M_{\eta^s}^\alpha, M_{\eta^{s^*}}^\alpha\} > \alpha M_h^\alpha$. The first condition is clear by the definition of η^s. The second condition is true because

$$\begin{aligned}
K M_{\eta^s}^\alpha (J+y)^\beta > M_h \quad \Leftrightarrow \quad M_{\eta^s}^\alpha > \frac{M_h}{K(J+y)^\beta} &= \frac{(\alpha K (J+y)^\beta)^{\frac{1}{1-\alpha}}}{K(J+y)^\beta} \\
&= \alpha^{\frac{1}{1-\alpha}} (K(J+y)^\beta)^{\frac{\alpha}{1-\alpha}} \\
&= \alpha \alpha^{\frac{\alpha}{1-\alpha}} (K(J+y)^\beta)^{\frac{\alpha}{1-\alpha}} \\
&= \alpha M_h^\alpha.
\end{aligned}$$

Thus, we have

$$\frac{\tau_t^{s^*}}{\tau_t^s} = \alpha^{\eta^{s^*}-\eta^s}\left(\frac{M_{\eta^{s^*}}}{M_{\eta^s}}\right)^\alpha > \alpha^{\eta^{s^*}-\eta^s}\left(\frac{\alpha M_h^\alpha}{M_h^\alpha}\right) = \left(\frac{\alpha}{\alpha^{\eta^s-\eta^{s^*}}}\right) > 1, \quad \text{since } \alpha < 1.$$

■

Proof of Proposition III.3 Using the implicit function theorem, we examine the sensitivity of I^* with respect to the parameters (S_M, S_T, y, w). Denoting $\Delta(I) = g(f(I)) - h(g(I))$, we have,

$$\frac{\partial I^*}{\partial S_M} = -\frac{\frac{\partial \Delta}{\partial S_M}}{\frac{\partial \Delta}{\partial I^*}}, \qquad \frac{\partial I^*}{\partial S_T} = -\frac{\frac{\partial \Delta}{\partial S_T}}{\frac{\partial \Delta}{\partial I^*}}, \qquad \frac{\partial I^*}{\partial y} = -\frac{\frac{\partial \Delta}{\partial y}}{\frac{\partial \Delta}{\partial I^*}}, \qquad \frac{\partial I^*}{\partial w} = -\frac{\frac{\partial \Delta}{\partial w}}{\frac{\partial \Delta}{\partial I^*}}. \tag{III.14}$$

Because $\frac{\partial \Delta}{\partial I} < 0 \ \forall I$ (Lemma III.3), the expressions (III.14) reduces to:

$$\frac{\partial I^*}{\partial S_M} > 0 \Leftrightarrow \frac{\partial \Delta}{\partial S_M} > 0, \quad \frac{\partial I^*}{\partial S_T} > 0 \Leftrightarrow \frac{\partial \Delta}{\partial S_T} > 0,$$

$$\frac{\partial I^*}{\partial y} > 0 \Leftrightarrow \frac{\partial \Delta}{\partial y} > 0, \quad \frac{\partial I^*}{\partial w} > 0 \Leftrightarrow \frac{\partial \Delta}{\partial c} > 0.$$

Note that $\frac{\partial I^*}{\partial x} > 0$ means it is optimal to *delay* the hiring until I_t reaches a higher amount. Moreover, we introduce the following notations to simplify the proofs:

$$\Pi_f \equiv KM_f^\alpha J^\beta - M_f, \quad \Pi_g(S_M, S_T) \equiv KM_g^\alpha (J - S_T)^\beta - M_g - S_M,$$

$$\Pi_h(y, w) \equiv KM_h^\alpha (J + y)^\beta - M_h - w.$$

We focus on the first case (i). The cases (ii) and (iii) follow the same logic. We will consider three exhaustive cases (Assumption III.1): (a) $M_g + S_M \le I < f(I)$, (a) $I \le M_g + S_M \le f(I)$, (c) $I \le f(I) \le M_g + S_M$. For all three cases, we will first evaluate $\frac{\partial}{\partial S_M} g(f(I))$, evaluate $\frac{\partial}{\partial S_M} h(g(I))$, and show that $\frac{\partial}{\partial S_M}\{g(f(I)) - h(g(I))\} \ge 0$.

(a) First, $g(f(I)) = f(I) + \Pi_g(S_M, S_T)$ if $f(I) \ge M_g + S_M$. Thus, we have $\frac{\partial}{\partial S_M} g(f(I)) = -1$.

If $I \geq M_g + S_M$ and $g(I) < M_h$, $h(g(I)) = K(I + \Pi_g(S_T, S_M))^{\alpha}(J+y)^{\beta} - w\}$, and hence $\frac{\partial}{\partial S_M} h(g(I)) = \alpha K(I + \Pi_g(S_T, S_M))^{\alpha-1}(J+y)^{\beta} \times (-1)$. Otherwise, if $I \geq M_g + S_M$ and $g(I) > M_h$, $h(g(I)) = I + \Pi_g(S_T, S_M) + \Pi_h(y, w)$, and hence $\frac{\partial}{\partial S_M} h(g(I)) = 0 + (-1) + 0 = -1$.

Since $\alpha K(I + \Pi_g(S_T, S_M))^{\alpha-1}(J+y)^{\beta} = \frac{\partial}{\partial x} K x^{\alpha}(J+y)^{\beta}\Big|_{x=I+\Pi_g(S_T,S_M)} \geq 1$, we have that $\frac{\partial}{\partial S_M}\{g(f(I)) - h(g(I))\} \geq 0$.

(b) First, $g(f(I)) = f(I) + \Pi_g(S_M, S_T)$ if $f(I) \geq M_g + S_M$. Thus, we have $\frac{\partial}{\partial S_M} g(f(I)) = -1$.

If $I \leq M_g + S_M$ and $g(I) < M_h$, $h(g(I)) = K(g(I))^{\alpha}(J+y)^{\beta} - w$, and hence $\frac{\partial}{\partial S_M} h(g(I)) = \alpha K(g(I))^{\alpha-1}(J+y)^{\beta} \times \{-\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta}\}$. Otherwise, if $I \leq M_g + S_M$ and $g(I) > M_h$, $h(g(I)) = K(I - S_M)^{\alpha}(J - S_T)^{\beta} + \Pi_h(y, w)$, and $\frac{\partial}{\partial S_M} h(g(I)) = \{-\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta}\}$.

Because $\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta} = \frac{\partial}{\partial x} g(x)\Big|_{x=I-S_M} \geq 1 \quad \Leftrightarrow \quad -\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta} \leq -1$, and $\alpha K(g(I))^{\alpha-1}(J+y)^{\beta} = \frac{\partial}{\partial x} h(x)\Big|_{x=g(I)} \geq 1$, we have $\frac{\partial}{\partial S_M}\{g(f(I)) - h(g(I))\} \geq 0$.

(c) First, if $f(I) < M_g + S_M$, $g(f(I)) = K(f(I) - S_M)^{\alpha}(J - S_T)^{\beta}$, and thus, $\frac{\partial}{\partial S_M} g(f(I)) = -\alpha K(f(I) - S_M)^{\alpha-1}(J - S_T)^{\beta}$.

If $I \leq M_g + S_M$ and $g(I) < M_h$, $h(g(I)) = K(g(I))^{\alpha}(J+y)^{\beta} - w$, and $\frac{\partial}{\partial S_M} h(g(I)) = \alpha K(g(I))^{\alpha-1}(J+y)^{\beta} \times \{-\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta}\}$. On the other hand, if $I \leq M_g + S_M$ and $g(I) > M_h$, $h(g(I)) = K(I - S_M)^{\alpha}(J - S_T)^{\beta} + \Pi_h(y, w)$, and $\frac{\partial}{\partial S_M} h(g(I)) = -\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta}$.

Since $f(I) > I$, $\frac{\partial}{\partial x} g(x)\Big|_{x=f(I)-S_M} < \frac{\partial}{\partial x} g(x)\Big|_{x=I-S_M}$, and hence $-\alpha K(f(I) - S_M)^{\alpha-1}(J - S_T)^{\beta} = -\frac{\partial}{\partial x} g(x)\Big|_{x=f(I)-S_M} > -\frac{\partial}{\partial x} g(x)\Big|_{x=I-S_M} = -\alpha K(I - S_M)^{\alpha-1}(J - S_T)^{\beta}$. Moreover, since $\alpha K(g(I))^{\alpha-1}(J+y)^{\beta} = \frac{\partial}{\partial x}h(x)\Big|_{x=g(I)} \geq 1$, $-\alpha K(f(I)-S_M)^{\alpha-1}(J-S_T)^{\beta} > -\alpha K(I-S_M)^{\alpha-1}(J-S_T)^{\beta} \times \{\alpha K(g(I))^{\alpha-1}(J+y)^{\beta}\}$. Thus, $\frac{\partial}{\partial S_M}\{g(f(I))-h(g(I))\} \geq 0$.

Now we focus on case (iv). We will prove for the case when $I^* < f(I^*) < M_g + S_M$ and $g(I^*) < M_h$. The remaining scenarios follow the same logic.

$$\begin{aligned}
\frac{\partial}{\partial S_T}g(f(I)) &= \frac{\partial}{\partial S_T}\{K(KI^{\alpha}J^{\beta}-S_M)^{\alpha}(J-S_T)^{\beta}\} \\
&= -\beta K(KI^{\alpha}J^{\beta}-S_M)^{\alpha}(J-S_T)^{\beta-1} = \left(-\frac{\beta}{J-S_T}\right)g(f(I)), \\
\frac{\partial}{\partial S_T}h(g(I)) &= \frac{\partial}{\partial S_T}\left\{K(K(I-S_M)^{\alpha}(J-S_T)^{\beta})^{\alpha}(J+y)^{\beta}-w\right\} \\
&= K(J+y)^{\beta}\cdot\alpha(K(I-S_M)^{\alpha}(J-S_T)^{\beta})^{\alpha-1}\cdot -\beta K(I-S_M)^{\alpha}(J-S_T)^{\beta-1} \\
&= \left(-\frac{\beta}{J-S_T}\right)\alpha K(K(I-S_M)^{\alpha}(J-S_T)^{\beta})^{\alpha}(J+y)^{\beta}.
\end{aligned}$$

$$\frac{\partial}{\partial S_T}\{g(f(I))-h(g(I))\} > 0$$

$$\begin{aligned}
\Leftrightarrow\quad & \left(\frac{\beta}{J-S_T}\right)\left\{\alpha K(K(I-S_M)^{\alpha}(J-S_T)^{\beta})^{\alpha}(J+y)^{\beta}\right\} \\
& \qquad -\left(\frac{\beta}{J-S_T}\right)\left\{K(KI^{\alpha}J^{\beta}-S_M)^{\alpha}(J-S_T)^{\beta}\right\} > 0 \\
\Leftrightarrow\quad & \alpha K(K(I-S_M)^{\alpha}(J-S_T)^{\beta})^{\alpha}(J+y)^{\beta} > K(KI^{\alpha}J^{\beta}-S_M)^{\alpha}(J-S_T)^{\beta} \\
\Leftrightarrow\quad & \frac{\alpha K^{\alpha}(I-S_M)^{\alpha^2}(J+y)^{\beta}}{(KI^{\alpha}J^{\beta}-S_M)^{\alpha}} > (J-S_T)^{\beta(1-\alpha)} \\
\Leftrightarrow\quad & S_T > J - \left(\frac{\alpha K^{\alpha}(I-S_M)^{\alpha^2}(J+y)^{\beta}}{(KI^{\alpha}J^{\beta}-S_M)^{\alpha}}\right)^{\frac{1}{\beta(1-\alpha)}}.
\end{aligned}$$

■

Proof of Corollary III.5 By Corollary III.4(i), I^* is such that $h_{\ell}(I^*) = h_{\ell+1}(I^*)$, i.e.

$$KI^{\alpha}(J+\sum_{j=1}^{\ell}y_j)^{\beta} - \sum_{j=1}^{\ell}w_j = KI^{\alpha}(J+\sum_{j=1}^{\ell+1}y_j)^{\beta} - \sum_{j=1}^{\ell+1}w_j.$$

Let $\Psi(I,J) \equiv KI^{\alpha}\left\{(J+\sum_{j=1}^{\ell+1} y_j)^{\beta} - (J+\sum_{j=1}^{\ell} y_j)^{\beta}\right\} - w_{\ell+1} = 0.$

Applying the implicit function theorem, we have that

$$\frac{\partial I^*}{\partial J} = -\frac{\partial\Psi(I,J)/\partial J}{\partial\Psi(I,J)/\partial I} = -\frac{\beta KI^{\alpha}\{(J+\sum_{j=1}^{\ell+1} y_j)^{\beta-1} - (J+\sum_{j=1}^{\ell} y_j)^{\beta-1}\}}{\alpha K\{(J+\sum_{j=1}^{\ell+1} y_j)^{\beta} - (J+\sum_{j=1}^{\ell} y_j)^{\beta}\}I^{\alpha-1}}$$

$$= \frac{-\beta I}{\alpha\{(J+\sum_{j=1}^{\ell+1} y_j)^{\beta} - (J+\sum_{j=1}^{\ell} y_j)^{\beta}\}}\left\{\frac{1}{(J+\sum_{j=1}^{\ell+1} y_j)^{1-\beta}} - \frac{1}{(J+\sum_{j=1}^{\ell} y_j)^{1-\beta}}\right\}.$$

Thus, we have $\frac{\partial I^*}{\partial J} > 0$ if and only if $\beta < 1$. ■

Proof of Lemma III.4.

$$\begin{aligned}
R(M,J) &= K(pJ^q + (1-p)M^q)^{r/q} \\
R'(M,J) &= K\left(\frac{r}{q}\right)(pJ^q + (1-p)M^q)^{r/q-1} \times q(1-p)M^{q-1} \\
R''(M,J) &= K\left(\frac{r}{q}\right)(pJ^q + (1-p)M^q)^{r/q-1} \times q(q-1)(1-p)M^{q-2} \\
&\quad + q(1-p)M^{q-1}K\left(\frac{r}{q}-1\right)\left(\frac{r}{q}\right)(pJ^q + (1-p)M^q)^{r/q-1} \\
&\quad \times q(1-p)M^{q-1} \\
&= K\left(\frac{r}{q}\right)(pJ^q + (1-p)M^q)^{r/q-1}q(1-p)M^{q-2} \\
&\quad \times\left(q-1+\left(\frac{r}{q}-1\right)q\frac{(1-p)M^q}{pJ^q+(1-p)M^q}\right).
\end{aligned}$$

The first term is positive since the q's cancel, and thus we have the expression is concave in M if and only if

$$\begin{aligned}
& q-1+\left(\frac{r}{q}-1\right) q \frac{(1-p)M^q}{pJ^q+(1-p)M^q} < 0 \\
\Leftrightarrow \quad & (\alpha+\beta-q)\left(\frac{(1-p)M^q}{pJ^q+(1-p)M^q}\right) < 1-q \\
\Leftrightarrow \quad & (\alpha+\beta)\left(\frac{(1-p)M^q}{pJ^q+(1-p)M^q}\right) < 1-q+q\left(\frac{(1-p)M^q}{pJ^q+(1-p)M^q}\right) \\
\Leftrightarrow \quad & (\alpha+\beta)\left(\frac{\alpha M^q}{\beta J^q+\alpha M^q}\right) < 1-q\left(\frac{\beta J^q}{\beta J^q+\alpha M^q}\right) \\
\Leftrightarrow \quad & (\alpha+\beta)\alpha M^q < \beta J^q+\alpha M^q-q\beta J^q \\
\Leftrightarrow \quad & (\alpha+\beta-1)\alpha M^q < (1-q)\beta J^q \\
\Leftrightarrow \quad & (\alpha+\beta-1)\left(\frac{\alpha}{\beta}\right)\left(\frac{M}{J}\right)^q < (1-q).
\end{aligned}$$

Clearly, (i) if $\alpha+\beta \leq 1$, then $R(M,J)$ is always concave in M because the left hand side is negative. Also (ii) if $q=0$ (Cobb Douglas), we have $R(M,J)$ is concave in M if $\alpha<1$ because

$$(\alpha+\beta-1)\left(\frac{\alpha}{\beta}\right) = \left(\frac{\beta-(1-\alpha)}{\beta}\right)\alpha < \alpha < 1.$$

(iii) If $q<0$ and $\alpha+\beta>1$, the left hand side of the inequality is decreasing in M, and thus the inequality holds and the $R(M,J)$ is concave for all M if $\frac{M}{J} \geq \left(\frac{(1-q)}{(\alpha+\beta-1)\left(\frac{\alpha}{\beta}\right)}\right)^{1/q}$. Finally, (iv) if $q \in (0,1]$, the left hand side is increasing in M and thus the $R(M,J)$ is concave in M for $\frac{M}{J} \leq \left(\frac{(1-q)}{(\alpha+\beta-1)\left(\frac{\alpha}{\beta}\right)}\right)^{1/q}$. ■

Proof of Corollary III.6. In the proof of Proposition 1, replace the expression $\frac{\alpha K(M^*)^\alpha (T^*)^\beta}{M^*}-1$ with $\frac{\partial \Pi(M^*,T^*)}{\partial M^*}$, $\frac{\beta K(M^*)^\alpha (T^*)^\beta}{T^*}$ with $\frac{\partial \Pi(M^*,T^*)}{\partial T^*}$, and $\frac{\alpha K(M^*)^\alpha (T^*)^\beta}{M^*}$ with $\frac{\partial R(M^*,T^*)}{\partial M^*}$ and the results follow. This is true as long as the structure of $\Pi(M,T) = R(M,T)-M$ remains. ■

Proof of Theorem III.3. From the expression of Corollary III.6, we have that

$$\frac{\mu^*_{t+1}}{\mu^*_t} = \left(\frac{\delta}{\frac{\partial R(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}}\right)\frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial M^*_t}\right)}, \quad \frac{\tau^*_{t+1}}{\tau^*_t} = \left(\frac{\delta}{\frac{\partial R(M^*_{t+1},T^*_{t+1})}{\partial T^*_{t+1}}}\right)\frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial T^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial T^*_t}\right)}.$$

Because $\Pi(M,J)$ is concave in M, and $M^*_{t+1} \geq M^*_t$, and because $\frac{\partial R_{t+1}}{\partial M^*_{t+1}} \geq 1$, we have

$$\left(\frac{\delta}{\frac{\partial R(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}}\right) \leq 1, \quad \frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial M^*_t}\right)} \leq 1.$$

Note that

$$\begin{aligned}\frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial T^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial T^*_t}\right)} &= \frac{\frac{r}{q}K(pM^q_{t+1}+(1-p)J^q)^{\frac{r}{q}-1}}{\frac{r}{q}K(pM^q_t+(1-p)J^q)^{\frac{r}{q}-1}}\frac{q(1-p)J^{q-1}}{q(1-p)J^{q-1}} \\ &= \left(\frac{pM^q_{t+1}+(1-p)J^q}{pM^q_t+(1-p)J^q}\right)^{\frac{r}{q}-1}.\end{aligned}$$

One can easily check that if $q < 0$ or if $q > 0, r > q$, the expression is greater than 1, hence satisfying $\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}\right) / \left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial M^*_t}\right) \leq 1 < \left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial T^*_{t+1}}\right) / \left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial T^*_t}\right)$, satisfying the proposition. Finally, for $q > 0, r < q < 1$, we can elaborate out the expressions for comparison:

$$\frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial T^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial T^*_t}\right)} = \left(\frac{\frac{r}{q}K(pM^q_{t+1}+(1-p)J^q)^{\frac{r}{q}-1}}{\frac{r}{q}K(pM^q_t+(1-p)J^q)^{\frac{r}{q}-1}}\right)\left(\frac{q(1-p)J^{q-1}}{q(1-p)J^{q-1}}\right).$$

$$\frac{\left(\frac{\partial \Pi(M^*_{t+1},T^*_{t+1})}{\partial M^*_{t+1}}\right)}{\left(\frac{\partial \Pi(M^*_t,T^*_t)}{\partial M^*_t}\right)} = \left(\frac{\frac{r}{q}K(pM^q_{t+1}+(1-p)J^q)^{\frac{r}{q}-1}-1}{\frac{r}{q}K(pM^q_t+(1-p)J^q)^{\frac{r}{q}-1}-1}\right)\left(\frac{qpM^{q-1}_{t+1}}{qpM^{q-1}_t}\right).$$

Comparing the second fractions inside the parenthesis for the respective expressions, we see that the top one is equal to 1 whereas the bottom one is less than 1. Similarly, comparing the first fractions inside the parenthesis for the respective expressions, we see that the bottom expression is smaller than the top expression because both numerator and denominator of bottom expression is greater than 0 and the fraction is less than

1, and adding 1 to both the numerator and the denominator of the bottom expression result in the top expression. Hence, the μ^* has a smaller half-life than τ^*. ■

The following Lemma is necessary for the proof of Theorem III.4.

Lemma III.5 *Let s_ℓ^m denote the optimal period to hire the ℓ^{th} employee, when hiring a total of m employees. Then, $s_1^1 = s_1^2$.*

Proof of Lemma III.5. We prove by contradiction. We will first show that (i) $s_1^2 > s_1^1$ is a contradiction, then show that (ii) $s_1^2 < s_1^1$ is a contradiction.

(i) Suppose that $s_1^1 < s_1^2$. Then, the capital available before hiring the second employee in period s_2^2 can be improved by setting $s_1^2 = s_1^1$. With increase in the available capital, the objective can be maximized, contradicting that s_1^2 is the optimal period to hire the first employee.

(ii) Suppose that $s_1^1 > s_1^2$. There are two potential cases: (a) when the first employee and the second employee are hired back to back periods (i.e. $\cdots \to h_0 \to g_1 \to g_2 \to h_2 \to \cdots$), and (b) when they are not hired back to back.

(a) We have

$$h_0 \to h_0 \to g_1 \to h_1 \to h_1 \quad \text{optimal for} \quad m = 1,$$
$$h_0 \to g_1 \to g_2 \to h_2 \to h_2 \quad \text{optimal for} \quad m = 2.$$

By the case of $m = 1$, $h_0(h_0(I)) > I_1^* > h_0(I)$, and by the case of $m = 2$, $g_1(h_0(I)) > I_2^*$. However, because by Assumption III.3, $\Pi_{g_1} \leq 0$, i.e. $g_1(I) < I$. Thus, we have $I_1^* > h_0(I) > g_1(h_0(I)) > I_2^* > I_1^*$, which is a contradiction.

(b) We have

$$h_0 \to h_0 \to g_1 \to h_1 \to h_1 \quad \text{optimal for} \quad m = 1,$$

$$h_0 \to g_1 \to h_1 \to g_2 \to h_2 \quad \text{optimal for} \quad m = 2.$$

The capital available in period s_2^2 when hiring the second employee can be improved by delaying the hiring of the first employee, because $h_1(g_1(h_0)) < g_1(h_0(h_0(I)))$. ■

Proof of Theorem III.4 The same argument used for proving Lemma III.5 can be used to show that $s_1^3 = s_1^2$, $s_1^4 = s_1^3$, etc. Then, once the optimal timing of the first employee is found, we can start in period $s_1^{\cdot}$ and set $m-1$ as the new m, and repeat the proof of Lemma III.5. We have our result by extending to all m. ■

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